MAA 3200 Dec 12, 2013
Final Exam and Key Prof. S. Hudson

Unlabeled problems are 10 points each.

1) [20 pts] Answer True or False. The universal set is the real numbers (with the usual operations, metric, etc).

S is a neighborhood of x if and only if x is an interior point of S.

S is open if and only if S' is closed.

$\forall f,\ S,\ T,\ f(S \cap T) = f(S) \cap f(T)$

Every open set has an interior point.

If $\forall n,\ A_n$ is open, then $\cap_{n=1}^{\infty} A_n$ is also open.

A finite set has no accumulation points and no interior points.

S is countable if and only if there exists a 1-1 function $f : S \to N$.

If x is a limit point of S, then it is also an accumulation point of S.

There is exactly one relation on $\{1,\ 2\}$ which is both symmetric and anti-symmetric.

If rng $\{x_n\}$ is finite, then $\{x_n\}$ has a Cauchy subsequence.

2) [5 pts] Compute $\cap_{n=1}^{\infty}(\frac{4n-1}{2n}, 5 + \frac{3}{n})$.

3) [5 pts] Let $S = \{1,\ 2,\ \ldots 10\}$. Define the usual equivalence relation $a \equiv b \pmod 3$ on S to mean $3|(a-b)$. List the elements of [4].

4) [5 pts] Let a be an accumulation point of dom(f). Write out *the negation* of the definition of $\lim_{x \to a} f(x) = L$. You can use abbreviations such as $\forall$ but avoid $\neg$ (or any equivalent).

5) [5 pts] List 10 consecutive integers, none of which are prime. Hint: why is a number like $8! + 7$ clearly not prime? For a little EC, prove that $\forall n, \exists a \in N$ such that none of $\{a, a+1, \ldots a+n\}$ are prime (perhaps on the back of the page).

6) [5 pts each] Let $f(x) = 3x^2 + 2$. 6a) Find $f^{-1}(f(3))$ taking care to use proper notation.

b) Using the same f, find the pre-image of $[1, 2]$.

7) [5 pts] Explain briefly why the complex numbers are not a complete ordered field. Do not use the uniqueness theorem for this.

8) [20 pts] State each result precisely.

Axiom of Choice

Schroder-Bernstein Thm.

Fundamental Thm of Arithmetic

Godel's Thm (this one does not have to be as precise)

Cantor's Thm

9) Find a closed formula for the sum of the first k odd integers, $1 + 3 + 5 + \dots (2k - 1)$. Prove it using induction.

10) [5 points] Briefly but precisely, define $Z = \{[(a, b)]\}$ from N. Then define $<$ on Z.

11) Choose ONE. Prove it as done in the text or lectures.

a) If $f : R \to R$ is continuous and $S \subset \text{rng } (f)$ is open, then $f^{-1}(S) \subset \text{dom } (f)$ is open.

b) State and prove the Intermediate Value Thm.

Bonus [approx 5 pts]: Prove: If $\liminf x_n = \limsup x_n$, then $\{x_n\}$ converges.

Remarks and Answers: The average was approx 59, with scores ranging from 40 to 65 (a fairly tight, slightly low range). I do not set a scale for the final, though it will be combined with the other grades, and the combined grades will then be scaled. The average grades on the various problems also varied from about 40% to 65% except that on problem 9 (induction), it was approx 95%. I was happy with the general level of *skills* shown on this exam, but not so much with problems 1, 8 and 11 which measured *study* of specific vocabulary, theorems and proofs.

1) TTFFF TTFFT

2) $[2, 5]$. Note that $\frac{4n-1}{2n} < 2 < 5 < 5 + \frac{3}{n}$ so the endpoints belong to every interval A_n. Also, note that the answer is NOT open (this example may help explain the 5th TF above).

3) $\{1, 4, 7, 10\}$

4) $\exists \epsilon > 0, \forall \delta > 0, \exists x \in \text{dom}(f), 0 < |x - a| < \delta$ and $|f(x) - f(a)| \geq \epsilon$.

5) $\{11!+2, 11!+3 \dots 11!+10, 11!+11\}$. It is not clear whether or not a number like 11!+1 is prime, so it is best to start with $11!+2$. For this same reason, it is better to use 11! (or 12!, etc) than 10!.

6a) $\{-3, 3\}$ 6b) $\{0\}$.

7) This number system has no partial order with trichotomy (because $i^2 = -1 < 0$). For full credit, I wanted to see the word *trichotomy*, or some similar reasoning.

8) See the text, lectures or the net. The results here were a bit weak, but you will be expected to learn this kind of thing in advanced courses (though you may not always be tested this way). Same thing for definitions! These are increasingly important in writing more advanced proofs, and you just have to put some effort into them.

9) k^2. It is a fairly standard induction proof (see the text) and most people did it well.

10) Let $A = N \times N$ and let $(a, b) \sim (c, d)$ mean $a + d = b + c$. Let $Z = A/\sim$. Let $[(a, b)] < [(c, d)]$ mean $a + d < b + c$.

11) See the text for a) and lecture notes for b).