MAA 3200 Dec 4, 2018
Final Exam Prof. S. Hudson

The universal set is the real numbers, if not specified.

1) Short answer (20 pts total). Show some work for each, but don't prove your answers.

a) Let $x_n = \frac{(-1)^n}{3} + 3 + \frac{3(-1)^n}{n}$. Find $\limsup x_n$.

b) Give an example of a set S and a 1-1 function $f : S \to S$ which is not onto.

c) Give an example of two functions f, g which are uniformly continuous on a set $S \subseteq R$ such that fg is not uniformly continuous there.

d) Let $I_x = (x - \frac{1}{x}, x + \frac{1}{x})$ for $x \in [1, 3]$. This family covers the set $S = [0.5, 3.1]$. Find a finite subcover.

e) Define the relation $x < y$ on R as done in class and in the online notes (using x_n, etc).

2) [20 pts total] Circle TWO of these and prove them.

a) The union of any two compact sets is compact (use the definition).

b) If f and g are continuous on R then $f + g$ is too (use the ϵ method).

c) $\lim_{x \to 3} 4x^2 = 36$ (use the ϵ method).

d) For each $n \in N$, let $S_n = \{p2^{-n} |\ p \in N\} \cap (\pi, +\infty)$. Let $x_n = \min\ S_n$. Show that x_n is a Cauchy sequence of rational numbers. Is $[x_n]_\sim$ bigger, smaller or equal to π (explain briefly, but do not prove this last part).

3) [15pts total] Use these relations on the set N to answer a,b and c below. Do not use other examples which are not listed here. If two of these are possible (for part (a) for example), you can just give one. If none of these work, say so - otherwise, you can answer with just one letter and you do not have to explain.

Let $S = \{(m, n)|\ m^3 - 3^m = n^3 - 3^n\}$. For example, $(0, 2) \in S$. [typo corrected, from (0,3), on 12/5/18]

Let $T = \{(n, n)|\ n \text{ is even}\}$.

Let mUn mean $m|n$ (m divides n).

Let mVn mean $m + n > 3$.

a) Give an example of a relation above that is symmetric and anti-symmetric.

b) Which of the relations above is an equivalence relation ?

c) Which is a partial order ?

4) Short answer [15 pts total]. In the definitions you may have to introduce a few variables such as x, S, ϵ, etc.

a) Define *boundary point.* Let $S = Q \cap (2, 3] \subseteq R$. Find the boundary points of S (viewed as a subset of R, as usual).

b) Define *accumulation point.* Let $S = Z \cup (2, 4) \subseteq R$. Find the accumulation points of S.

c) Define *interior point.* Let $S = \{(x, y) \subseteq R^2 |\ xy \geq 0$. Using either of the metrics from class, find the interior points of S.

5) [20 pts] Answer True or False. You do not have to explain.

$\neg(\neg P \to \neg Q) \Leftrightarrow \neg P \wedge Q$

$(\exists k \in Z, 2k = 6) \wedge (\exists j \in Z, 3j = 13)$

If R is a reflexive relation on $A = \{2,\ 3\}$, then R is transitive.

If S, R are relations on a set A, then Dom $(S \circ R) =$ Dom (R)

Thomae's function is continuous at every rational number in $(0, 1)$.

$\exists! x \in R, (x^2 - 9 = 0 \wedge (3 - x)^3 > 3)$

$A \subseteq B$ iff $P(A) \subseteq P(B)$.

If $A \subseteq C$, and $B \cap C = \emptyset$, then $A \subseteq B^c$.

If $f : R \to R$ is continuous and $A \subseteq R$, then $f^{-1}(A)$ is equinumerable with A.

The set $S = \{(x, y) \subseteq R^2 |\ xy \geq 0\}$ is open.

6) [10 pts] Choose one textbook proof. State the theorem more precisely if needed.
a) Every Cauchy sequence of real numbers converges.
b) If f is continuous on R, then the pre-image of every open set is also open.
c) The intermediate-value theorem.
d) If a sequence is monotone and bounded then it converges.

Bonus [about 4-8 pts]: This is a theorem which we did not cover in class, called the Nested Interval Theorem. Let $I_n = [a_n, b_n]$ be non-empty closed intervals in R. Assume they are *nested*, meaning $I_{n+1} \subseteq I_n$ for $n = 1, 2, 3, \ldots$. a) Prove that $\cap_{n=1}^{\infty} I_n$ is not empty. b) Give an example to show this is false for open intervals. c) Give another example to show this is false in Q (with closed intervals).

Remarks and Answers: The average was 68 with high scores of 98 and 91. The results were lower on the short answer problems 3 and 4 (approx 50%) and higher on the textbook proof, problem 6 (over 90%). I do not set a separate scale for the final exam, only for the semester total.

1a) 10/3. The last term approaches zero and the first terms have a repeated combined max of 10/3.

1b) One example is $S = N$ and $f(n) = n+1$. Since 0 is not in the range, this is not onto. I also accepted the slightly obscure but equivalent answer $f = \sigma$ (referring to the definition of N). There are many examples, but in all of them S has to be infinite.

1c) $f(x) = g(x) = x$ with $S = R$. Again, there are many other answers.

1d) I_x with $x = 1, 2, 2.5$ and 3 works (partly because $\frac{1}{2.5} + \frac{1}{3} > 3 - 2.5$). If you don't want to think so hard, let $x = 1, 1.01, 1.02 \ldots 3.0$ instead.

1e) $\exists N, \exists \epsilon > 0$ (in Q) if $n \geq N$ then $x_n + \epsilon < y_n$. See the online notes for the context.

2) Most popular was (c), then (b), then (a). Nobody chose (d).

The idea of (a) is simple, to take the union of two finite subcovers.

Most people who chose (b) did OK with the triangle inequality and the $\epsilon/2$ steps, but there were many gaps. Ideally your proof should include phrases such as - Fix $a \in R$ (and this comes before the $\epsilon > 0$ step), Let $\delta =$min (δ_1, δ_2), and a sequence near the end like $|(f(x) + g(x)) - (f(a) + g(a))| \leq \ldots \leq \epsilon$.

For (c), you can set $\delta =$ min $(1, \epsilon/28)$, for example. The most common gap was an explanation of $|x + 3| < 7$.

Part (d) was a simplified version of an assigned exercise about R.

3) a) T, b) S, c) U. The results were a bit low, maybe because small logical mistakes mattered. The comment that $(0, 2) \in S$ was a hint that S is not anti-symmetric. Notice that T not reflexive, but it is contained in the identity relation, so it is symmetric and anti-symmetric. Divisibility is a standard example of a partial order.

4) See the text for the definitions (which need to be precise). For the rest, with some examples which were not required;

a) $[2, 3]$. Ex: $2 \in \partial S$ because every $(2 - \epsilon, 2 + \epsilon)$ contains points in S (such as 2.0001) and in S^c (such as 2).

b) $[2, 4]$. Ex: 2 is one, because every $(2-\epsilon, 2+\epsilon)$ contains points in S (such as 2.0001).

c) Int $(S) = \{(x, y) \subseteq R^2 |\ xy > 0$. Ex: $P(3, 0) \in \partial(S)$ so it is not an interior point. Every $B_\epsilon(P)$ contains points in S (such as (3, 0.001)) and points in S^c (such as 3, -0.001)). This also shows the last TF in problem 5 is False.

Often the difference between open and closed comes down to the difference between $>$ and $\geq$ in a defining formula, though this is a rather superficial note, and maybe not very reliable.

5) TFTFF TTTFF For 4, consider $S = \emptyset$. For 9, consider the constant function $f(x) = 1$.

6) The most popular choice was (a) , then (b), and the results on both were good.

Bonus: a) Only one student made progress on this, but they did a great job. Outline of proof: The sequence a_n is bounded and monotone, so it converges to some L. Then explain that $\forall n, a_n \leq L \leq b_n$, so L is in the intersection. Done. The student used the BW.Thm. which is not really needed. Notice the similarity with our other existence proofs; you first have to construct L somehow, usually using some previous existence theorem, and then "justify it".

b) $I_n = (0, 1/n)$.

c) $I_n = \{x \in Q^+ : |x^2 - 2| \leq 1/n\}$.