MAA 3200 Dec. 10, 2003
Final Exam and Key Prof. S. Hudson

The average score was about 52/100, including 2 scores over 90, about 9 in [40, 90], and 5 below 40. The average scores were especially low on problems 2, 4, 6 and 7.

1) Prove that $\sum_{k=1}^{n} k(k+1) = n(n+1)(n+2)/3$ for $n \geq 1$, using induction.

Answer: Basis: Set $n = 1$ and check: $2 = 2$. Induction Step: Let $n \geq 1$ and assume the statement is true. Add $(n+1)(n+2)$ to both sides and get $\sum_{k=1}^{n+1} k(k+1) = n(n+1)(n+2)/3 + 3(n+1)(n+2)/3 = (n+1)(n+2)(n+3)/3$. This proves the statement for $n+1$.

2) Suppose that $E \subset R$ is a nonempty bounded set and sup $E \notin E$. Prove there is a strictly increasing sequence $\{x_n\}$ in E such that $x_n \to$ sup E.

Answer: Let sup $E = s$. We will construct a strictly increasing sequence in E such that $s - 1/n < x_n < s$, which implies $x_n \to s$. By the approximation theorem, there is an $x_1 \in E$ such that $s-1 < x_1 \leq s$. Actually, $x_n < s$, since $s \notin E$. Let $n > 1$ and assume that x_{n-1} has been defined successfully. By the same reasoning, there is an $x_n \in E$ such that max $\{x_{n-1}, s - 1/n\} < x_n < s$. This process defines a sequence that is strictly increasing, with $s - 1/n < x_n < s$, as desired. The Squeeze theorem shows that $x_n \to s$.

This kind of proof requires some planning at the start. You need to stop and think about how to construct the sequence x_n. It is likely you'll use an existence theorem (repeatedly) to do that, and hopefully the "sup E" reminds you of the approximation theorem. The inequality in that theorem might then remind you of the Squeeze theorem, and give you the idea of setting $\epsilon = 1/n$. To make x_n strictly increasing requires a modification of this basic idea. This was HW problem 2.3.2.

3) [20pts] Answer True or False: You don't have to explain.

Every bounded sequence has a convergent subsequence.

If limsup $x_n \leq$ liminf $x_n = K \in R$, then x_n converges to K.

limsup $a_n + b_n$ = limsup a_n + limsup b_n

If R is a symmetric relation on A then $R \circ R \subseteq R$.

There is a bijection $f : (0, 1) \to (0, 1]$.

N is equinumerable with Q.

If F is an ordered field, then F is isomorphic to R.

$(P \vee Q) \wedge (\neg P \vee \neg Q)$ is a contradiction.

$\forall \epsilon > 0, \exists \delta > 0, \forall x \in R, 0 < x < \delta \to 0 < 2x < \epsilon$

If $A \subseteq R$ and $f(x) = \chi_A(x)$, then $f^{-1}(\{1\}) = A$.

Answer: TTFFT TFFTT

4) Choose ONE:

A) State and prove the Extreme Value Theorem.

B) State and prove the Nested Interval Theorem.

C) Prove that every sequence has a monotone subsequence.

Answer: Parts A) and B) are in the text, and are on the review sheet. For C), define the nested sequence of sets, $T_n = \{x_n, x_{n+1}, \ldots\}$. There are two cases;

1) Suppose one of the T_n does not contain its supremum (so it does not have a greatest term). Then there is an increasing subsequence x_{n_k} inside that T_n. [The proof resembles that of problem 2 above. Just be sure to make $n_k > n_{k-1}$].

2) Otherwise, each of the T_n contains a greatest term. Let $x_{n_1} = \max T_1$. Let $x_{n_2} = \max T_{n_1+1} \leq \max T_{n_1} = x_{n_1}$. Continue in this manner, to get a decreasing (nonincreasing) subsequence.

5) [20pts] Recall that $a \equiv b \pmod{m}$ means $m|(a-b)$, and the equivalence classes are denoted Z_m. For this problem, you can use well-known formulas like $x < x+1$, $1/7 \in Q \setminus Z$, $\sqrt{2} \in R \setminus Q$, $x^2 \geq 0$, etc, without proof.

a) Prove this relation is symmetric on Z.

Answer: Assume $a \equiv b \pmod{m}$. So, $m|(a-b)$. Since $b - a = (-1)(a-b)$, this implies $m|(b-a)$, so $b \equiv a \pmod{m}$.

b) Addition and multiplication can be defined on Z_5 such that Z_5 is a field. Show that Z_6 (operations defined the same way) is not a field.

Answer: Since [2][3]=[6]=[0], the element [2] does not have a multiplicative inverse (and explain why, or check the 6 options).

c) Explain why the complex numbers cannot be made into an ordered field, no matter how $<$ is defined on it.

Answer: Since $i^2 = -1 < 0$, this would contradict the theorem that in an ordered field $\forall x,\ x^2 \geq 0$ [or, explain that trichotomy fails for $x = i$ and $y = 0$].

d) Explain why Q is not a *complete* ordered field (This doesn't have to be a careful proof, but include the definition of complete in your answer).

Answer: If Q were complete, every nonempty bounded set E would have a supremum. But let $E = \{x : x^2 < 2\}$. In R the supremum would be $\sqrt{2}$, but that's not in Q. Any rational number is either too big or too small to be the supremum [a careful proof of that might use density and the fact that x^2 is increasing on R].

6) Suppose $f : R \to R$ is continuous at a, and $f(a) > M$ for some $M \in R$. Prove there is an open interval I containing a such that $f(x) > M$ on I.

Answer: Apply the sign-change lemma 3.28 to $g(x) = f(x) - M$, which is positive and continuous at a. This is basically HW problem 3.3.4. You can also use an ϵ approach as in the proof of the lemma.

7) Give an example of each -

a) A continuous function $f : R \to R$ and an open set $A \subset R$ such that $f(A)$ is not open.

Answer: Let $f(x) = x^2$ and $A = (-1, 1)$, so $f(A) = [0, 1)$ [or let f be a constant function and $A = R$, or etc].

b) A sequence with $\liminf a_n = 0$ that does not converge.

Answer: Let $a_n = (-1)^n + 1$. Or, write it out as $0, 2, 0, 2, 0, \ldots$.

8) Prove: If $f : A \to B$ and $S, T \subseteq A$ then $f(S \cup T) \subset f(S) \cup f(T)$.

Answer: See the key to Exam II.

BONUS [5pts]: Choose at most ONE:

A) Prove that $B_1(0)$ is an *open* subset of R^2. You can assume that distance in R^2 satisfies the triangle inequality. I'd prefer that you do not include a picture, except maybe to guide yourself.

B) Prove one step of trichotomy on R: that $x < y$ and $x > y$ cannot both be true (use the $x = [\{x_n\}]$ notation, and known results in Q).

Answer to A: Let $x \in B_1(0)$. ETS x is an interior point. Since $|x| < 1$ we can set $\epsilon = 1 - |x| > 0$. ETS $B_\epsilon(x) \subseteq B_1(0)$. Let $y \in B_\epsilon(x)$ which means $|y - x| < \epsilon$. By the triangle inequality, $|y - 0| \leq |y - x| + |x - 0| < \epsilon + (1 - \epsilon) = 1$. This proves $y \in B_1(0)$. Done.

Remark: After the 2nd sentence, you could draw a picture of the ball $B_1(0)$ with the point x somewhere inside. Then draw a tiny ball $B_\epsilon(x)$ that is centered at x and stays inside the larger ball. The largest possible radius is $\epsilon = 1 - |x|$, which explains the third sentence of the proof. The rest of the proof is 'just checking'.

Such a picture is very useful as a guide in writing the proof, or even in explaining it, but don't rely too heavily on it. Don't draw conclusions from it that you can not justify. Pictures can be deceiving (at best, such a picture represents a single example which might - or might not - be typical). Note: the algebraic proof given above works in R^3 and R^4 and other settings, where a picture is impossible.

Of course, there are times when 'a picture is worth a thousand words' and we'll use them with some caution. [For example, cardinality proofs with arrows, the proof that $\sin x/x \to 1$, etc).

Answer to B: We did something like this in class using x_n's and ϵ's. See the Morash book on reserve for more about proofs like this one.