MAA 3200, Key to HW 3

As the problems get harder, don’t forget the basics such as - Remember to state your assumptions clearly.
Each assumption should be based on a valid proof strategy and should usually be followed by a new ”ETS”.
Explain any unusual or indirect strategy briefly - for example, with a phrase like ”to get a contradiction”.
If you get really stuck, ask for help.

I graded 3.5.2 and Limit C) for 20 points each, 4.1.5 and 6, 4.2.2a and 2b, for 10 each, and another 20 points overall. If you didn’t try Limit C), I graded 4.3.9 instead (but with a maximum of 10 points).

3.5.2 - Assume $A\Delta B = A\setminus B \cup B \setminus A \subseteq A$. ETS $B \subseteq A$. Let $x \in B$. ETS $x \in A$. Assume that $x \notin A$ to get a contradiction (using cases, $x \in A$ or $x \notin A$, is also OK). Then $x \in B \setminus A \subseteq A \setminus B \cup B \setminus A \subseteq A$. This contradiction proves $x \in A$.

Many people who tried this one never stated their assumptions clearly. Another common mistake was to focus too much on the definition of $A\Delta B$, while ignoring the goal, $B \subseteq A$. Some people erroneously concluded from $A\Delta B \subseteq A$ that $x \in A$. This problem was somewhat difficult, and the average grade on it was fairly low. The next four problems were worth 10 points each and most answers were OK.

4.1.5 - There should be four cases, including the cases $(x, y) \in A \times D$ and $(x, y) \in B \times C$.

4.1.6 - $|A \times B| = mn$.

4.2.2a - $L^{-1} \circ L = \{(s, t) \in S \times S \mid \exists r \in R, (s, r) \in L \text{ and } (r, t) \in L^{-1}\}$, which means the students $s$ and $t$ live in the same room, $r$.

4.2.2b - $E \circ (L^{-1} \circ L) = \{(s, c) \in S \times C \mid \exists t \in S, (s, t) \in L^{-1} \circ L \text{ and } (t, c) \in E\}$, which means the student $s$ lives with a student $t$ who is taking course $c$.

Limit C) This is a moderately hard problem, but the proof is fairly
similar to that of Problem B) of HW2, and to the related proof mentioned in the hint. However this related proof involves \( n \to \infty \) which means it uses \( N \) instead of \( \delta \) (compare the two definitions of limit). You should use \( \delta \)'s in this proof.

**Plan:** we can make \( f(x) \) arbitrarily close to \( L \) and to \( M \), so \( L \) and \( M \) must also be arbitrarily close to each other. If \( L \neq M \) this wouldn’t happen.

**Proof:** Assume the 2 limits are true, and that \( L \neq M \), to get a contradiction. Let \( \epsilon = |L - M| \). Choose \( \delta_1 \) so that \( 0 < |x - a| < \delta_1 \) implies \( |f(x) - L| < \epsilon/3 \) (\( \delta_1 \) exists since the first limit is true). Likewise, choose \( \delta_2 \) so that \( 0 < |x - a| < \delta_2 \) implies \( |f(x) - M| < \epsilon/3 \).

Next we need to introduce an \( x \) that makes all this true, so let \( x = a + \delta/2 \) where \( \delta = \min(\delta_1, \delta_2) \). Check that \( |x - a| \) is less than both \( \delta_1 \) and \( \delta_2 \).

The triangle inequality helps get the contradiction:

\[
\epsilon = |L - M| = |f(x) - M - (f(x) - L)| \\
\leq |f(x) - M| + |f(x) - L| \\
\leq \epsilon/3 + \epsilon/3
\]

This contradiction proves that \( L = M \).

Don’t worry too much if you didn’t get this one. Keep trying, and look for patterns in proofs from the text, lectures and answer keys. For example, in the proof above:

a) The triangle inequality is used very often with absolute value signs. Get used to it!

b) Min’s are used a lot, especially to make 2 things happen.

c) If a limit is a given, you’ll probably use it to produce a \( \delta \) to be used elsewhere (but first you must introduce a good \( \epsilon \)).

d) The smaller you choose your Greek letters, the better.
Some answers/hints to ungraded problems

3.5.1a: Use cases on this one (or, you can use a tautology instead). Let $x \in A \cap (B \cup C)$. So $x \in A$ and $x \in B \cup C$. So, $x \in B$ or $x \in C$.

Case 1: Suppose $x \in B$. Then $x \in A \cap B$, so $x \in (A \cap B) \cup C$.
Case 2: Suppose $x \in C$. Then $x \in (A \cap B) \cup C$. Done.

3.5.5. [This has 2 parts, and each part has two cases. Here’s one. The other 3 are fairly similar.] Proof of $\leftarrow$: Assume $|x - 4| > 2$. Case 1: Assume $x - 4 > 0$, so that $x - 4 = |x - 4| > 2$. So $x > 6$. So $x + x > x + 6$. So $2x - 6 > x$. Also $2x - 6 > 0$ so that $|2x - 6| > x$. Done with this case.

3.7.5 Prove that if $\lim_{x \to c} f(x) = L$ and $L > 0$ then $\exists \delta > 0$ such that $\forall x$, if $0 < |x - c| < \delta$ then $f(x) > 0$.

Plan of proof: This is good practice with the definition of limit and with quantifiers. It might be a good idea to draw a graph or two, including $f$, $c$, $L$, $\epsilon$ and $\delta$. But, as usual, the proof will finally depend on definitions and standard proof strategies. Looking over the problem quickly, we see that we’re going to have to produce a $\delta$ somehow.

The assumption $\lim_{x \to c} f(x) = L$ means we can pick a specific $\epsilon$ (like $\epsilon = 5$ or $\epsilon = L/2$ or whatever) and get back a $\delta$ ”that works”. [If picking an $\epsilon$ doesn’t seem right to you, see line 6, page 306, about how to use $\forall x, P(x)$]. It is not clear yet how to pick $\epsilon$, so let’s look a little deeper at what $\delta$ does, and at our ultimate goal, $f(x) > 0$.

This $\delta$ promises us an inequality $|f(x) - L| < \epsilon$. Using algebra to remove the annoying absolute value signs, we get $-\epsilon < f(x) - L < \epsilon$. Adding $L$ we get $L - \epsilon < f(x) < L + \epsilon$. This inequality implies that $f(x) > 0$ if $\epsilon = L$.

This may not be a clear convincing proof yet, but we have a choice of $\epsilon$ that seems promising, and we can start a careful proof. I am going to leave some of the routine steps to you.

Outline of the Proof: Assume that $\lim_{x \to c} f(x) = L$ and $L > 0$. Set $\epsilon = L$. The definition of limit implies that there is a $\delta > 0$ so that $\exists \delta > 0$ such that $|x - c| < \delta$ then $f(x) > 0$ if $\epsilon = L$.

ETS $f(x) > 0$. [fill the rest in]. Done.
3.7.6 Prove that if \( \lim_{x \to c} f(x) = L \) then \( \lim_{x \to c} 7f(x) = 7L \).

**Idea of Proof:**

Assume: \( \forall \epsilon_1 > 0, \exists \delta_1 > 0 \) etc

ETS: \( \forall \epsilon_2 > 0, \exists \delta_2 > 0 \) etc

In this exercise, we’ll get \( \delta_1 \) from the assumption, and (trust me here) we can set \( \delta_2 = \delta_1 \). But the \( \epsilon \)'s must be different! We’ll let \( \epsilon_2 \) be arbitrary. We could set \( \epsilon_1 = \epsilon_2 \), but it doesn’t work out. The key to picking \( \epsilon_1 \) is in this simple algebra:

\[
|7f(x) - 7L| < \epsilon_2 \Leftrightarrow |f(x) - L| < \epsilon_2 / 7
\]

This suggests setting \( \epsilon_1 = \epsilon_2 / 7 \). I leave the rest (which should be fairly routine) to you.

4.1.4a: Again, I suggest cases (see the proof of 4, pg 160). Let \((x, y) \in A \times (B \cup C)\). So, \( x \in A \) and \( y \in B \cup C \). So, \( y \in B \) or \( y \in C \).

Case 1: If \( y \in B \) then \((x, y) \in A \times B \subseteq (A \times B) \cup (A \times C)\). Done.

Case 2: If \( y \in C \), the proof is similar.

4.1.8: It is a little easier to prove the contrapositive because that will make it easy to introduce elements to discuss: Assume that \( A \) and \( C \) are not disjoint (so \( \exists x \in A \cap C \)), and that \( B \) and \( D \) are not disjoint (so \( \exists y \in B \cap D \)). Then \((x, y) \in A \times B \) and \((x, y) \in C \times D \). So \( A \times B \) and \( C \times D \) are not disjoint.

4.3.7a: (the \( \rightarrow \) part) Assume \( R \) is reflexive on \( A \), and that \( p \in i_A \).

Then \( p = (x, x) \) for some \( x \in A \). Since \( R \) is reflexive, \((x, x) \in R\), so \( p \in R \).

Done.

4.3.9a (very brief proofs): \( R \cap S \) is reflexive because each \((x, x)\) belongs to \( R \) and to \( S \), therefore to \( R \cap S \). Same for \( R \cup S \). Since \((x, x) \in R\), we get \((x, x) \in R^{-1} \) (from the def of \( R^{-1} \)), so yes.

\( R \circ S \) is too: Let \( x \in A \). ETS \( \exists b \in A, (x, b) \in S \) and \((b, x) \in R\). Set \( b = x \). Since \((x, x) \in R\) and \((x, x) \in S\), we get \((x, x) \in R \circ S \).