MAA 3200, Key to HW 3

As the problems get harder, don't forget the basics such as -
Remember to state your assumptions clearly.
Each assumption should be based on a valid proof strategy and should usually be followed by a new "ETS".
Explain any unusual or indirect strategy briefly - for example, with a phrase like "to get a contradiction".
If you get really stuck, ask for help.

I graded 3.5.2 and Limit C) for 20 points each, 4.1.5 and 6, 4.2.2a and 2b, for 10 each, and another 20 points overall. If you didn't try Limit C), I graded 4.3.9 instead (but with a maximum of 10 points).

3.5.2 - Assume $A \triangle B = A \setminus B \cup B \setminus A \subseteq A$. ETS $B \subseteq A$. Let $x \in B$. ETS $x \in A$. Assume that $x \notin A$ to get a contradiction (using cases, $x \in A$ or $x \notin A$, is also OK). Then $x \in B \setminus A \subseteq A \setminus B \cup B \setminus A \subseteq A$. This contradiction proves $x \in A$.

Many people who tried this one never stated their assumptions clearly. Another common mistake was to focus too much on the definition of $A \triangle B$, while ignoring the goal, $B \subseteq A$. Some people erroneously concluded from $A \triangle B \subseteq A$ that $x \in A$. This problem was somewhat difficult, and the average grade on it was fairly low. The next four problems were worth 10 points each and most answers were OK.

4.1.5 - There should be four cases, including the cases $(x, y) \in A \times D$ and $(x, y) \in B \times C$.

4.1.6 - $|A \times B| = mn$.

4.2.2a - $L^{-1} \circ L = \{(s, t) \in S \times S \mid \exists r \in R, (s, r) \in L \text{ and } (r, t) \in L^{-1}\}$, which means the students s and t live in the same room, r.

4.2.2b - $E \circ (L^{-1} \circ L) = \{(s, c) \in S \times C \mid \exists t \in S, (s, t) \in L^{-1} \circ L \text{ and } (t, c) \in E\}$, which means the student s lives with a student t who is taking course c.

Limit C) This is a moderately hard problem, but the proof is fairly similar to that of Problem B) of HW2, and to the related proof mentioned in the hint. However this related proof involves $n \to \infty$ which means it uses N instead of δ (compare the two definitions of limit). You should use δ's in this proof.

Plan: we can make $f(x)$ *arbitrarily close* to L and to M, so L and M must also be *arbitrarily close* to each other. If $L \neq M$ this wouldn't happen.

Proof: Assume the 2 limits are true, and that $L \neq M$, to get a contradiction. Let $\epsilon = |L - M|$. Choose δ_1 so that $0 < |x - a| < \delta_1$ implies $|f(x) - L| < \epsilon/3$ (δ_1 exists since the first limit is true). Likewise, choose δ_2 so that $0 < |x - a| < \delta_2$ implies $|f(x) - M| < \epsilon/3$.

Next we need to introduce an x that makes all this true, so let $x = a + \delta/2$ where $\delta = \min(\delta_1, \delta_2)$. Check that $|x - a|$ is less than both δ_1 and δ_2.

The triangle inequality helps get the contradiction:

$$\begin{aligned} \epsilon = |L - M| =& |f(x) - M - (f(x) - L)| \\ \leq& |f(x) - M| + |f(x) - L)| \\ \leq& \epsilon/3 + \epsilon/3 \end{aligned}$$

This contradiction proves that $L = M$.

Don't worry too much if you didn't get this one. Keep trying, and look for patterns in proofs from the text, lectures and answer keys. For example, in the proof above:

a) The triangle inequality is used very often with absolute value signs. Get used to it!

b) Min's are used a lot, especially to make 2 things happen.

c) If a limit is a *given*, you'll probably use it to produce a δ to be used elsewhere (but first you must introduce a good ϵ).

d) The smaller you choose your Greek letters, the better.

Some answers/hints to ungraded problems

3.5.1a: Use cases on this one (or, you can use a tautology instead). Let $x \in A \cap (B \cup C)$. So $x \in A$ and $x \in B \cup C$. So, $x \in B$ or $x \in C$.

Case 1: Suppose $x \in B$. Then $x \in A \cap B$, so $x \in (A \cap B) \cup C$.

Case 2: Suppose $x \in C$. Then $x \in (A \cap B) \cup C$. Done.

3.5.5. [This has 2 parts, and each part has two cases. Here's one. The other 3 are fairly similar.] Proof of $\leftarrow$: Assume $|x-4| > 2$. Case 1: Assume $x - 4 > 0$, so that $x - 4 = |x - 4| > 2$. So $x > 6$. So $x + x > x + 6$. So $2x - 6 > x$. Also $2x - 6 > 0$ so that $|2x - 6| > x$. Done with this case.

3.7.5 Prove that if $\lim_{x \to c} f(x) = L$ and $L > 0$ then $\exists \delta > 0$ such that $\forall x$, if $0 < |x - c| < \delta$ then $f(x) > 0$.

Plan of proof: This is good practice with the definition of limit and with quantifiers. It might be a good idea to draw a graph or two, including f, c, L, ϵ and δ. But, as usual, the proof will finally depend on definitions and standard proof strategies. Looking over the problem quickly, we see that we're going to have to produce a δ somehow.

The assumption $\lim_{x \to c} f(x) = L$ means we can pick a specific ϵ (like $\epsilon = 5$ or $\epsilon = L/2$ or whatever) and get back a δ "that works". [If picking an ϵ doesn't seem right to you, see line 6, page 306, about how to **use** $\forall x, P(x)$]. It is not clear yet how to pick ϵ, so let's look a little deeper at what δ does, and at our ultimate goal, $f(x) > 0$.

This δ promises us an inequality $|f(x) - L| < \epsilon$. Using algebra to remove the annoying absolute value signs, we get $-\epsilon < f(x) - L < \epsilon$. Adding L we get $L - \epsilon < f(x) < L + \epsilon$. This inequality implies that $f(x) > 0$ if $\epsilon = L$.

This may not be a clear convincing proof yet, but we have a choice of ϵ that seems promising, and we can start a careful proof. I am going to leave some of the routine steps to you.

Outline of the Proof: Assume that $\lim_{x \to c} f(x) = L$ and $L > 0$. Set $\epsilon = L$. The definition of limit implies that there is a $\delta > 0$ so that [fill this in]. Let x be arbitrary and assume [fill this in]. ETS $f(x) > 0$. [fill the rest in]. Done.

3.7.6 Prove that if $\lim_{x \to c} f(x) = L$ then $\lim_{x \to c} 7f(x) = 7L$.

Idea of Proof:

Assume: $\forall \epsilon_1 > 0, \exists \delta_1 > 0$ etc

ETS: $\forall \epsilon_2 > 0, \exists \delta_2 > 0$ etc

In this exercise, we'll get δ_1 from the assumption, and (trust me here) we can set $\delta_2 = \delta_1$. But the ϵ 's must be different! We'll let ϵ_2 be arbitrary. We could set $\epsilon_1 = \epsilon_2$, but it doesn't work out. The key to picking ϵ_1 is in this simple algebra:

$$|7f(x) - 7L| < \epsilon_2 \Leftrightarrow |f(x) - L| < \epsilon_2/7$$

This suggests setting $\epsilon_1 = \epsilon_2/7$. I leave the rest (which should be fairly routine) to you.

4.1.4a: Again, I suggest cases (see the proof of 4, pg 160). Let $(x, y) \in A \times (B \cup C)$. So, $x \in A$ and $y \in B \cup C$. So, $y \in B$ or $y \in C$.

Case 1: If $y \in B$ then $(x, y) \in A \times B \subseteq (A \times B) \cup (A \times C)$. Done.

Case 2: If $y \in C$, the proof is similar.

4.1.8: It is a little easier to prove the contrapositive because that will make it easy to introduce elements to discuss: Assume that A and C are not disjoint (so $\exists x \in A \cap C$), and that B and D are not disjoint (so $\exists y \in B \cap D$). Then $(x, y) \in A \times B$ and $(x, y) \in C \times D$. So $A \times B$ and $C \times D$ are not disjoint.

4.3.7a: (the $\rightarrow$ part) Assume R is reflexive on A, and that $p \in i_A$. Then $p = (x, x)$ for some $x \in A$. Since R is reflexive, $(x, x) \in R$, so $p \in R$. Done.

4.3.9a (very brief proofs): $R \cap S$ is reflexive because each (x, x) belongs to R and to S, therefore to $R \cap S$. Same for $R \cup S$. Since $(x, x) \in R$, we get $(x, x) \in R^{-1}$ (from the def of R^{-1}), so yes.

$R \circ S$ is too: Let $x \in A$. ETS $\exists b \in A, (x, b) \in S$ and $(b, x) \in R$. Set $b = x$. Since $(x, x) \in R$ and $(x, x) \in S$, we get $(x, x) \in R \circ S$.