MAA 3200, Key to HW 4

I graded 4.6.10a, 5.1.1abc, 6.1.2, and 7.1.7b for 20 points each, plus 20 overall. More answers on page 2. The average was about 70/100.

4.6.10a: (reflexive) Let $a \in Z$. Note $m \cdot 0 = a - a$, so $a \equiv a \pmod{m}$.

(symmetric) Assume $a \equiv b \pmod{m}$. So, $mk = a - b$ for some $k \in Z$. So, $m(-k) = b - a$. Since $-k \in Z$, this proves $b \equiv a \pmod{m}$.

5.1.1:

a) Yes, each $a \in A$ is related to 4 and only to 4.
b) No. The 1 is related to more than one $b \in B$.
c) No. Nothing is related to d.

6.1.2: Be sure to include enough words in your induction proofs, even if they seem to be mostly calculations.

Proof: We use induction on n. Basis step: Clearly $0^2 = 0 \times 1 \times 1/6$. Induction step: Let $n \geq 0$. Assume that $0^2 + \ldots n^2 = n(n+1)(2n+1)/6$. ETS $0^2 + \ldots n^2 + (n+1)^2 = (n+1)(n+2)(2n+3)/6$. Add $(n+1)^2$ to both sides of the assumption: $0^2 + \ldots n^2 + (n+1)^2 = n(n+1)(2n+1)/6 + (n+1)^2 = etc = (n+1)(n+2)(2n+3)/6$. Done. [I skipped some algebra, but everybody got that part if they tried].

7.1.7b: You can't use a "common-sense" formula like $|B| < |A|$ here, unless you can justify it. That's what exercises 7.1.5 through 7.1.7a are about. You can use these in your proof of 7.1.7b.

Proof: Assume that $B \subset A$, $B \neq A$, so there is some $a \in A$, with $a \notin B$. Assume $A \sim B$. Assume that A is finite (to get a contradiction). By 7.1.5b, there is a unique $n \in N$ such that A has n elements. This means there is a bijection $g : A \to \{1, 2, \ldots n\}$.

Since $a \notin B$ and g is 1-1, $g(a) \notin g(B)$. So, $g(B) \subset \{1, 2, \ldots n\}$ and is not equal. So, by the second part of 7.1.7a, $g(B)$ has fewer than n elements. By 7.1.6 and $A \sim B$, B is finite and has n elements. But $B \sim g(B)$, so by 7.1.6 again, $g(B)$ has n elements, a contradiction.

7.1.3a: (not graded) No. For example, set $A = B = N$ and let $C = \{a\}$ and $D = \{a, b\}$. Both $A \times C$ and $B \times D$ are denumerable.

7.1.3b: (not graded) No. Use the same sets as in 3a above.

7.1.7a: Use induction on n. Basis: If $n = 0$ then $A = \emptyset$ and A is finite with zero elements (the "empty function" proves that). Induction: Let $n \geq 0$ and assume 7.1.7a is true for n (this is the "IH"). Assume $A \subseteq \{1, 2, \ldots n{+}1\} = S$. If $A = S$, then clearly A is finite with $n{+}1$ elements and 7.1.7a is true. If $A \neq S$, there is some $j \notin A$, with $1 \leq j \leq n + 1$. [if $j = n + 1$, we can quote the IH and stop. We can't assume this, but watch...]. It is easy to construct a bijection $f : S \to S$ such that $f(j) = n + 1$. So, $f(A) \subseteq \{1, 2, \ldots n\}$ and we can apply the IH to $f(A)$. So, $f(A)$ is finite and has at most n elements. Since $A \sim f(A)$, we can say the same about A, by 7.1.6, which proves 7.1.7a (with n replaced by $n + 1$).