MAA 3200, Key to HW 5

For HW5, I graded Special 3), 1.1.3, 1.2.7 and 1.4.9a for 20 points each, plus 20 overall. The average was about 60/100, which is low.

Try to understand all these problems before the next exam. As always, you are welcome to ask me for hints before the HW is due, or for complete answers after it is due. You can come by my office or email me.

Special 3) [Show trichotomy for Z] Proof: Let $x, y \in Z$, so $x = [(a,b)]$ and $y = [(c,d)]$ where $a, b, c, d \in N$.

Case 1: $a + d < b + c$. This is the definition of $x < y$.
Case 2: $b + c < a + d$. This is the definition of $y < x$.
Case 3: $a + d = b + c$. This is the definition of $(a,b) \sim (c,d)$ which is equivalent to $x = y$.

By trichotomy in N exactly one of these cases is true. This proves trichotomy in Z.

1.1.3 Assume $a \leq b$ which means $a < b$ or $a = b$. Case 1: If $a < b$ then $a + c < b + c$ by the additive property. Case 2: If $a = b$ then $a + c = b + c$ [this doesn't have to be explained. If $a = b$ then one can be substituted for the other]. In either case, $a + c \leq b + c$.

1.2.7 Assume $0 \leq a < b$. I will leave the case $a = 0$ to you and assume $0 < a$.

1) Ets $\forall n, 0 \leq a^n < b^n$. We'll use induction on n.

BASIS: Let $n = 1$ (the first element of N in Wade's book). Ets: $0 \leq a^1 < b^1$ but we assumed this.

Ind. Step: Let $n \geq 1$ and assume $0 \leq a^n < b^n$. Ets $0 \leq a^{n+1} < b^{n+1}$. Since $a > 0$, $0 \leq a^{n+1} < ab^n$. Since $b^n > 0$, $ab^n < b^{n+1}$. By transitivity, $a^{n+1} < b^{n+1}$. Done with 1).

2) Ets $\forall n, 0 \leq a^{1/n} < b^{1/n}$. I don't think induction works here, but an indirect approach does. If $b^{1/n} < a^{1/n}$ then $b < a$ by 1) (contradiction). I leave some details to you (the 0 and the = case).

1.4.9a: I graded the 1-1 part. Almost all answers were good.

Comments on some of the ungraded problems:

Vell 7.2.1 - This is done in Wade's book (in one sentence).

7.2.6 - Use Thm 7.2.4, with $A =$ the characters on a keyboard.

Special 2) - ETS $\forall m \in N, 0 + m = m + 0$. By defn, $m + 0 = s_m(0)$ and $0 + m = s_0(m)$. By a) $s_m(0) = m$. So, ETS $\forall m \in N, s_0(m) = m$. We'll use induction.

Basis: Let $m = 0$ (we are still assuming that N starts at 0 as in Velleman). ETS: $s_0(0) = 0$, but this follows from a) [setting $m = 0$].

Induction: Let $m \in N$ and assume $s_0(m) = m$. ETS $s_0(\sigma(m)) = \sigma(m)$. [This is the version of induction to use until we have defined $m+1$ and proven some properties of addition. Maybe you could use "$m + 1$" instead of $\sigma(m)$ already, but that seems harder]. Using b), we get $s_0(\sigma(m)) = \sigma(s_0(m))$, which by hypothesis is $\sigma(m)$. Done.

Special 4a) - We are given that $1 \neq 0$. So, by trichotomy, $0 < 1$ (and we are done) or $1 < 0$. If $1 < 0$ then multiplying this by 1 implies $0 < 1$ (by the multiplicative property) and we are done. [There's no need to point out the contradiction!]. There are probably lots of other proofs (eg Example 1.2 of Wade).

Special 4b) - Assume F has a least element x, and show that $x - 1$ is a smaller element of F.

Wade 1.2.2a) - Set $a = b = 1$ in the binomial theorem.

1.3.3 - The simplest proof is to use cardinality. Since (a, b) is uncountable and Q is countable, $(a, b)\backslash Q$ is uncountable and therefore is not empty.

Here's another proof (without using Wade Ch 1.4), which is also simple, but it took me a quite while to find it. By thm 1.24, we know $\exists x \in Q \cap (a + \sqrt{2}, b + \sqrt{2})$. This implies that $y = x - \sqrt{2} \in (a, b)$ and $y \notin Q$.

1.3.4 - Just apply Thm.1.24 to the interval $(a - 1/n, a + 1/n)$.

1.3.6a) - Prove the "not unique" part by giving a simple example. Prove the "unique" part as done in Remark 1.19. You shouldn't use Theorem 1.28 because it assumes uniqueness of the infimum (circular reasoning).

1.4.6 - This is a fairly long problem. One proof strategy is to show $a \to b \to c \to d \to a$, which has 4 parts. In each part, state your assumption and goal clearly.