MAA 3200, Key to HW 6

I graded 2.1.1a, 2.1.6a, 2.2.5 and 2.3.2 for 20 points each, plus 20 overall. The average was about 70/100.

2.1.1a: Show $3 + 1/n \to 3$. Proof: Let $\epsilon > 0$. Choose an integer $N > 1/\epsilon$ (it exists by the Archimedian Property, but we've explained that so often, I think it can now be omitted). Assume $n \geq N$. So, $1/n \leq 1/N < \epsilon$. So, $|(3 + 1/n) - 3| = |1/n| = 1/n < \epsilon$ and we are done.

2.1.6a: Assume $x_n \to L$ and $y_n \to L$. ETS $x_n - y_n \to 0$. Let $\epsilon > 0$. From the assumptions, we know $\exists N_x,\ n \geq N_x \to |x_n - L| < \epsilon/2$ and $\exists N_y,\ n \geq N_y \to |y_n - L| < \epsilon/2$. Let $N = \max\{N_x, N_y\}$. Assume $n \geq N$. Then $|(x_n - y_n) - 0| = |x_n - L + L - y_n| \leq |x_n - L| + |L - y_n| < \epsilon/2 + \epsilon/2 = \epsilon$. Done.

Remark: If you didn't get full credit for this one, study the organization of the proof above carefully. There's not much flexibility. For example, you can't write these sentences in another order. Also, you can write out the definition of $x_n - y_n \to 0$, as a guide to follow when making the proof.

2.2.1: Try to use the Squeeze theorem in all these, since using the definition of limit will be messier. For example, **2.2.1d:** Prove $n/2^n \to 0$.

Proof: Since we know $1/n \to 0$, ETS $0 \leq n/2^n \leq 1/n$ for all large n. So, ETS $n^2 \leq 2^n$ for $n \geq 5$. This is Example 6.1.3 in Velleman (it's an induction proof).

2.2.5: Let $x \in R$ (ETS $\exists r_n \to x$). Let $n \in N$. By the density theorem, there is a rational number $r_n \in I_n = (x - 1/n, x + 1/n)$. Since $1/n \to 0$, the endpoints of I_n converge to x. By the Squeeze theorem, so does r_n.

Remark: This short proof does require some planning. If you have to prove a sequence r_n exists, then you have to define r_n for each n. Often you'll use formulas with $1/n$ in them, and maybe an existence theorem, to define your r_n. Compare with 2.3.2 below.

A picture of a number line might help plan out this proof. Also, I hope you saw the need for rational numbers near x, and remembered the density theorem. Work on making associations like that. Read over each theorem carefully before trying the HW, and try to decide what it is about.

2.3.2: Let $s = \sup E \notin E$. We will define x_n recursively (I'll explain why later). Let $x_1 \in E$ be arbitrary. Let $n \geq 2$ and assume $x_{n-1} \in E$ has been defined. Set $a_n = \max\{x_{n-1}, s - 1/n\}$. Note that since $x_{n-1} \in E$ and $s \notin E$ these two points aren't equal. So, $a_n < s$. By the approximation property, there is a number $x_n \in E$ such that $a_n < x_n \leq s$. Since $0 < s - a_n \leq 1/n$ we know $a_n \to s$. By the Squeeze theorem, $x_n \to s$, too. Since $x_n > a_n \geq x_{n-1}$, the sequence is strictly increasing. Done.

Remark: If the problem did not require "strictly increasing", the proof would be much simpler, and we could set $a_n = s - 1/n$. But since x_n must be related to x_{n-1}, we have to use recursion and be fairly careful.

2.3.11a: We will prove $\forall n \in N, y_{n-1} \leq y_n \leq x_n \leq x_{n-1}$, which proves both sequences are monotone. It also shows both are bounded above by x_0 and below by y_0. Then by the Monotone Convergence theorem, they both converge. To prove the claim, we use induction. I leave the basis step ($n = 1$) to you, and begin with the induction hypothesis,

$$\text{Assume } y_{n-1} \leq y_n \leq x_n \leq x_{n-1}$$

$$\text{ETS} \quad y_n \leq y_{n+1} \leq x_{n+1} \leq x_n$$

which has three parts. For the third, $x_{n+1} = x_n(y_n + y_n)/(x_n + y_n) \leq x_n$ because $y_n \leq x_n$. Similarly, $x_{n+1} = (x_n + x_n)y_n/(x_n + y_n) \geq y_n$. Using this, we also get $y_{n+1} = \sqrt{x_{n+1}y_n} \geq y_n$, which is the first part. Using it again, we get $y_{n+1} = \sqrt{x_{n+1}y_n} \leq x_{n+1}$ which is the second part. This completes the induction step, and the proof.

2.3.11b: Square both sides of the definition of y_n and take limits to get $y^2 = xy$, which implies $y = x$. We know $y_n < x < x_n$ for all n. So, if we compute a few terms of each sequence, that should prove the last inequality of the problem.