MAA 3200, Key to HW 8, 12/4/03

I graded 3.1.1a, 3.2.7a, and 3.3.1 for 20 points each, plus 40 overall. The average was about 65/100.

2.1.1a: Prove $\lim_{x\to 2} x^2 - x + 1 = 3$.

Let $\epsilon > 0$. Set $\delta = \min\{1, \epsilon/4\}$ [scratch work omitted]. Assume $0 < |x-2| < \delta$. So, $|x-2| < \epsilon/4$. And $|x-2| < 1$ which implies $|x| < 3$ and $|x+1| < 4$ [algebra]. So, $|(x^2 - x + 1) - 3| = |(x-2)(x+1)| < (\epsilon/4)4 = \epsilon$. Done.

Remark: This form of the proof is probably the shortest, and the easiest to read. But other forms are OK, and may be easier to write. For example, the second sentence could be "WLOG $\delta \leq 1$." followed by some scratch work, including the [algebra] line above. If you choose this second style, you should probably put more effort into the explanation, and shouldn't refer the reader to your scratch work (redo it, with justifications).

3.2.7a: Assume $f(x) \geq g(x)$ and $\lim_{x\to a} g(x) = +\infty$. ETS $\lim_{x\to a} f(x) = +\infty$. Using the definition, let $M \in R$ be given. Since $g \to \infty$, there is a $\delta > 0$ such that if $0 < |x-a| < \delta$ then $g(x) > M$. With the same δ and the same assumption on x, we get $f(x) \geq g(x) > M$. Done.

Remark: I don't see any other way to do this one, because most of our comparison theorems are stated only for limits that converge in R (eg $L \neq \infty$). Theorem 3.10 assumes the two functions converge in R. Likewise, using sequential convergence is a good idea, but Theorem 2.17 also assumes convergence in R. The main point of this exercise is to show that these theorems can be extended to the $L = \infty$ case. Also, it is not OK to assume $f(x) \to L \in R$ (planning to get a contradiction), because that is not the correct negation of the ETS.

3.3.1: Prove that $\exists x \in R$, $e^x = x^2$.

Set $f(x) = e^x - x^2$, which is continuous on R (by the comment in the text). Note $f(0) = 1 - 0 > 0$ and $f(-1) = 1/e - 1 < 0$. By the IVThm, there is an $x \in (-1, 0)$ such that $f(x) = 0$, so that $e^x = x^2$.

Remark: Do *not* try to solve for x here. Since this exercise is a $\exists x$ statement involving continuous functions, it should remind you of the $\exists x$ theorem(s) in this section. If it didn't, spend more time with the theorems before starting the HW !

The choice of f above is a pretty standard trick for reducing a problem from the study of two functions to the study of one function. It is not clear what we are allowed to assume about e^x, I think we have to assume that $e^0 = 1$ and $e^{-1} < 1$, so I did.