MAA 3200 HW on Number Systems; Fall 2018

Here are answers to problems 4, 5 and 6 per a request.

4) Prove that every nonzero element of Q (as defined in class and in the online file) has a multiplicative inverse. You should suggest an equivalence class and justify that it works.

Proof: Let $x \neq 0$ in Q. This means $x = [(a, b)]$ with $a, b \neq 0$ in Z. Let $y = [(b, a)]$. Then $y \in Q$ because $a \neq 0$. And $xy = [(ab, ba)] = 1$. So $y = 1/x$. Done.[1]

5) Before defining + on N, we defined (for each fixed $m \in N$) a function $s_m : N \to N$ by:

a) $s_m(0) = m$

b) $s_m(\sigma(n)) = \sigma(s_m(n))$

. . . [some remarks omitted, see nzq.pdf] . . . so ETS: $\forall m \in N,\ s_0(m) = m$.

Proof that $\forall m \in N,\ s_0(m) = m$. We use induction on m.

Basis) Let $m = 0$. Then $s_0(m) = s_0(0) = 0$ by the definition (part a).

Induction) Fix $m \in N$. Assume $s_0(m) = m$. ETS $s_0(\sigma(m)) = \sigma(m)$.[2] But $s_0(\sigma(m)) = \sigma(s_0(m)) = \sigma(m)$ by the definition (part b) and the IH. Done.

6) Prove trichotomy for $<$ on Z, using trichotomy for $<$ on N. Of course, you will need to use the definition of $<$ on Z and may have to think about what $=$ means for Z. Your proof will probably use cases.[3]

[1]I have assumed the reader is familiar with the definitions of 0 and 1, and that $yx = xy$. Otherwise a few more steps could be included.

[2]I feel this ETS should be included in every induction proof, though this proof is so short, it might be omitted.

[3]In hindsight, this exercise is a little simpler than I expected, and cases are not required. I am slightly uncomfortable using the phrase "this means" in a proof, as it often burdens the reader, but in this example it seems clearer than cases would be.

Proof: Let $x = [(a, b)]$, and $y = [(c, d)] \in Z$. By the trichotomy principle in N, either $a+d < b+c$ or $a+d > b+c$ or $a+d = b+c$ (and only one of these is true). By the definitions (of $<$, etc) this means $x < y$ or $x > y$ or $x = y$ (and only one of these is true). Done.