MAA 3200, Fall 2018, Construction of the Real Numbers
see also Ch. 9.3 of Morash, on reserve

This will complete our plan to construct four major number systems in the order $N \to Z \to Q \to R$. We have constructed Q and proven or accepted its familiar properties, that it is an ordered field for example. We now construct R and discuss its properties:

Thm: R is a complete ordered field (review Kane Ch. 2.5 for the meaning).

Thm: If F is a complete ordered field then F is isomorphic to R.

Again, we will not prove everything, but will include completeness, density and perhaps trichotomy. The next steps use Cauchy sequences heavily. We will use some theorems from Kane, which hold in both Q and R. The exception is the theorem that Cauchy sequences converge (in R but not in Q). That depended on the Completeness Axiom for R, which we have not proven yet, so using it now would be circular reasoning. I plan to go over the basic definitions and the easy properties rather quickly, then pay more attention to a few selected theorems.

Definitions: Let A be the set of all Cauchy sequences $\{x_n\}$ of rational numbers. For example, $3, 3.1, 3.14, \ldots$ is in A. This sequence will correspond to π but so will other sequences in A such as $3, 17, 3.1, 3.14, \ldots$ The first 1000 terms don't really count. Next, we define an equivalence relation $\sim$ on A. Let $\{x_n\} \sim \{y_n\}$ mean $\lim(x_n - y_n) = 0$ (defined as usual, except that everything, including ϵ, is in Q). Now define

$$R = A/\sim$$

So, for example, we will think of π as the equivalence class of the sequence $3, 3.1, 3.14, \ldots$ (and of the other sequence above). We can regard a rational number such as $2/3$ as a real number defined by a constant sequence, $2/3 = [\{2/3\}]$. We can define addition by $[\{x_n\}] + [\{y_n\}] = [\{x_n + y_n\}]$. Notice that $\{x_n + y_n\} \in A$ because we know Q is closed under addition and that the sum of two Cauchy sequences is also Cauchy (the same reasoning works in both Q and R). As usual, one should ideally check that + is well-defined, but that is mostly a technicality. The other operations are similar. For example, $[\{x_n\}] \cdot [\{y_n\}] = [\{x_n \cdot y_n\}]$ with similar comments. Division requires more thought, but if $y = [\{y_n\}] \neq 0$, we can prove that for large enough n we get $y_n \neq 0$ and then x/y eventually makes sense.

The definition of $x < y$ is a little bit awkward. You might expect something like $x_n < y_n$ for all n (or maybe for all large n) but that does not work out well. For example, $0 < [\{1/n\}]$ is false (they are equal), even though every $0 < 1/n$. The correct definition uses a small rational number $\epsilon > 0$.

Definition: $[\{x_n\}] < [\{y_n\}]$ means $\exists \epsilon > 0, \exists N$ if $n > N$ then $x_n + \epsilon < y_n$.

Notice that if $x, y \in Q$ then $x < y$ has the same meaning in R as in Q. You can also check some basics, that if $x < y$ then $0 < y - x$ and $\exists \delta > 0, x + \delta < y$ and the transitivity, and so on. We do not have time to prove all these, but see Morash if interested. Define $\leq$ as the union of $<$ and $=$, and also define least upper bound in the usual way.

Thm (Completeness): If $\emptyset \neq S \subset R$ has an upper bound $M \in R$ then it has a least upper bound $L \in R$.

Proof: Fix S and M. WLOG S contains a positive number and $M \in N$, so that $M = [\{M\}]$. By the well-ordering principle, we can assume M is the smallest upper bound among the whole numbers. We will construct a Cauchy sequence of upper bounds $x_n \in Q$ such that $L = [\{x_n\}] \in R$ is the lub of S. Let $x_0 = M$. Let $x_1 = M - 2^{-1}$ if that is an upper bound of S, otherwise let $x_1 = M$. Repeat; let $x_{n+1} = x_n - 2^{-n-1}$ if that is an upper bound of S, otherwise let $x_{n+1} = x_n$.

Exercise 1: Prove that this defines a Cauchy sequence $\{x_n\} \in Q$.

Exercise 2: Let $s \in R$. Prove that if $s \leq x_n$ for all n, then $s \leq L$.

Exercise 3: Prove (perhaps by induction) that $x_n - 2^{-n}$ is never an upper bound of S.

These complete the proof of the theorem. Exercise 1 shows $L \in R$. Exercise 2 shows L is an upper bound of S. Exercise 3 shows no $K < L$ is an upper bound of S ($K < L$ implies $\exists \epsilon, K < L - \epsilon$ and $\exists n, K < x_n - 2^{-n}$, so K is not an upper bound). So $L = \text{lub} S$.

Thm (Density): Q is *dense* in R; if $a < b$ in R then $\exists q \in Q, a < q < b$.

Proof: By the Archimedean principle, there is some $n \in N$ with $1/n < b-a$. There is some minimal $m \in Z$ such that $nb \leq m$. Then $q = (m-1)/n \in (a, b)$.

Exercise 4: The previous proof has some small gaps. Fill them (explain each step as needed).

Exercise 5: Use this theorem to prove that the irrationals are also dense in R. Hint: it is easy to construct an irrational number between two rationals such as 3 and 4; for example $x = 3 + (4-3)/\sqrt{2}$.

These last theorems finally justify our reasoning in several proofs over the last few months. If time permits, I may also insert a proof of trichotomy here, or in the last lectures. Here are a few more exercises related to fields or R, for additional practice.

6) Suppose F is a field and $x, y \in F$ and $xy = 0$. Prove that $x = 0$ or $y = 0$. [Note: if $x \in F$ and $x \neq 0$, then $\exists a \in F, ax = xa = 1$].

7) Prove that + is well-defined on R. (Assume that $[a'] = [a]$ and $[b'] = [b]$. ETS: $[a'] + [b'] = [a] + [b]$).

8) Explain why Z_6 is not a field.

9) Prove that it is not possible to define $<$ on the complex numbers, to make it an ordered field. [Assume it is. Use the fact that $\exists i \in C$, $i^2 = -1$, and trichotomy, to get a contradiction.] Do you think Z_5 can be made into an ordered field? Explain.

10) Prove that the $\sim$ used to define R is an equivalence relation.

11) Prove that addition is commutative in R (you can assume it is in Q).

12) Prove that $<$ is transitive on R from the definition of $<$.