MAD2104

Suggested problems on Chapter 1 material
(Logic)

1. Let p and q be the following propositions.
p: It is below freezing.
q: It is snowing.
Write the following propositions using p and q and logical connectives (including negations).
a. It is below freezing and snowing.
b. It is below freezing but not snowing.
c. It is not below freezing and it is not snowing.
d. It is either snowing or below freezing (or both).
e. If it is below freezing, it is also snowing.
f. Either it is below freezing or it is snowing, but it is not snowing if it is below freezing.
g. That it is below freezing is necessary and sufficient for it to be snowing.

a. $p \wedge q$

b. $p \wedge \neg q$

c. $(\neg p) \wedge (\neg q)$ which can also be written $\neg p \wedge \neg q$

d. $q \vee p$ (which is equivalent to $p \vee q$)

e. $p \to q$

f. $(p \vee q) \wedge (p \to \neg q)$

g. $\underbrace{(q \to p)}_{\substack{p \text{ is necessary for } q \\ \text{(to have } q\text{, you must have } p\text{)}}} \wedge \underbrace{(p \to q)}_{\substack{p \text{ is sufficient for } q \\ (p \text{ is enough to guarantee } q)}}$ or $p \longleftrightarrow q$

2. Let p, q, and r be the following propositions.
p: Grizzly bears have been seen in the area.
q: Hiking is safe on the trail.
r: Berries are ripe along the trail.
Write the following propositions using p, q, and r and logical connectives (including negations).

a. Berries are ripe along the trail, but grizzly bears have not been seen in the area.

b. Grizzly bears have not been seen in the area and hiking on the trail is safe, but berries are ripe along the trail.

c. If berries are ripe along the trail, hiking is safe if and only if grizzly bears have not been seen in the area.

d. It is not safe to hike on the trail, but grizzly bears have not been seen in the area and the berries along the trail are ripe.

e. For hiking on the trail to be safe, it is necessary but not sufficient that berries not be ripe along the trail and for grizzly bears not to have been seen in the area.

f. Hiking is not safe on the trail whenever grizzly bears have been seen in the area and berries are ripe along the trail.

a. $r \wedge \neg p$

b. $((\neg p) \wedge q) \wedge r$ or $(\neg p) \wedge q \wedge r$ or $\neg p \wedge q \wedge r$

c. $r \rightarrow (q \leftrightarrow \neg p)$

d. $(\neg q) \wedge ((\neg p) \wedge r)$ or $(\neg q) \wedge (\neg p) \wedge r$ or $\neg q \wedge \neg p \wedge r$

e. $\underbrace{q \rightarrow ((\neg r) \wedge (\neg p))}$ AND ... ? See next page

For q to be true, it is necessary to have both $\neg r$ and $\neg p$

f. $((\neg p) \wedge r) \rightarrow (\neg q)$ or $(\neg p \wedge r) \rightarrow \neg q$

More comments on 2e:

Consider a related example.

Let s be the statement "You run fast"

Let t be the statement "You win an Olympic sprinting medal"

Then the translation of

"To win an Olympic sprinting medal, it is necessary that you run fast" would be: $t \rightarrow s$.

(If you're told that someone won an Olympic sprinting medal, then you can conclude that they run fast.)

What about "To win an Olympic sprinting medal, it is not sufficient to run fast"?

This says: It's <u>not</u> true that running fast implies winning an Olympic sprinting medal.

That can be written as $\neg(s \rightarrow t)$.

However, maybe a better translation would be "There exists at least one person who runs fast but has not won an Olympic sprinting medal." That would use quantifiers.

Anyway, 2e is a trickier than average question.

3. Construct a truth table for each of these compound propositions.

a. $p \to \neg p$
b. $p \leftrightarrow \neg p$
c. $p \oplus (p \vee q)$
d. $(p \wedge q) \to (p \vee q)$
e. $(p \vee q) \to (p \oplus q)$
f. $((p \to q) \wedge (q \to r)) \to (p \to r)$
g. $(p \oplus q) \oplus (p \oplus \neg q)$

a.

p	$\neg p$	$p \to \neg p$
1	0	0
0	1	1

b.

p	$\neg p$	$p \leftrightarrow \neg p$
1	0	0
0	1	0

c.

p	q	p	$\oplus$	$(p \vee q)$
1	1	1	0	1
1	0	1	0	1
0	1	0	1	1
0	0	0	0	0

d.

p	q	$(p \wedge q)$	$\to$	$(p \vee q)$
1	1	1	1	1
1	0	0	1	1
0	1	0	1	1
0	0	0	1	0

e.

p	q	$(p \vee q)$	$\to$	$(p \oplus q)$
1	1	1	0	0
1	0	1	1	1
0	1	1	1	1
0	0	0	1	0

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f.

p	q	r	$((p \to q) \wedge (q \to r)) \to (p \to r)$				
1	1	1	1	1	1	$1 \to 1 = 1$	1
1	1	0	1	0	0	$0 \to 0 = 1$	0
1	0	1	0	0	1	$0 \to 1 = 1$	1
1	0	0	0	0	1	$0 \to 0 = 1$	0
0	1	1	1	1	1	$1 \to 1 = 1$	1
0	1	0	1	0	0	$0 \to 1 = 1$	1
0	0	1	1	1	1	$1 \to 1 = 1$	1
0	0	0	1	1	1	$1 \to 1 = 1$	1

antecedent

consequent

final answer in this column

g.

p	q	$(p \oplus q) \oplus (p \oplus \neg q)$				
1	1	0	1	1	1	0
1	0	1	1	1	0	1
0	1	1	1	0	0	0
0	0	0	1	0	1	1

4. Show that each of these conditional statements is a tautology by using truth tables.

a. $(p \wedge q) \to p$
b. $p \to (p \vee q)$
c. $\neg p \to (p \to q)$
d. $(p \wedge q) \to (p \to q)$
e. $\neg(p \to q) \to p$
f. $\neg(p \to q) \to \neg q$

a.

p	q	$(p\wedge q)$	$\to$	p
1	1	1	1	1
1	0	0	1	1
0	1	0	1	0
0	0	0	1	0

b.

p	q	p	$\to$	$(p \vee q)$
1	1	1	1	1
1	0	1	1	1
0	1	0	1	1
0	0	0	1	0

c.

p	q	$\neg p$	$\to$	$(p \to q)$
1	1	0	1	1
1	0	0	1	0
0	1	1	1	1
0	0	1	1	1

d.

p	q	$(p\wedge q)$	$\to$	$(p \to q)$
1	1	1	1	1
1	0	0	1	0
0	1	0	1	1
0	0	0	1	1

e.

p	q	$\neg$	$(p \to q)$	$\to$	p
1	1	0	1	1	1
1	0	1	0	1	1
0	1	0	1	1	0
0	0	0	1	1	0
		antecedent			consequent

f.

p	q	$\neg$	$(p \to q)$	$\to$	$\neg q$
1	1	0	1	1	0
1	0	1	0	1	1
0	1	0	1	1	0
0	0	0	1	1	1
		antecedent			consequent

5. Show that each conditional statement in Question 4 is a tautology **without** using truth tables.

a. $(p \wedge q) \rightarrow p \equiv \neg(p \wedge q) \vee p$ using rule $X \rightarrow Y \equiv \neg X \vee Y$

$\equiv (\neg p \vee \neg q) \vee p$ DeMorgan

$\equiv \neg p \vee \neg q \vee p$ associativity

$\equiv \neg p \vee p \vee \neg q$ commutativity

$\equiv (\neg p \vee p) \vee \neg q$ associativity

$\equiv T \vee \neg q$ $\neg X \vee X \equiv T$

$\equiv T$ $T \vee X = T$

b. $p \rightarrow (p \vee q) \equiv \neg p \vee (p \vee q)$ rule $X \rightarrow Y \equiv \neg X \vee Y$

$\equiv \neg p \vee p \vee q$

$\equiv (\neg p \vee p) \vee q$

$\equiv T \vee q$

$\equiv T$

5c. $\neg p \to (p \to q) \equiv \neg\neg p \vee (p \to q)$ $\quad X \to Y \equiv \neg X \vee Y$

$$\equiv p \vee (p \to q)$$
$$\equiv p \vee (\neg p \vee q)$$
$$\equiv (p \vee \neg p) \vee q$$
$$\equiv T \vee q$$
$$\equiv T$$

d. $(p \wedge q) \to (p \to q) \equiv \neg(p \wedge q) \vee (p \to q)$

$$\equiv (\neg p \vee \neg q) \vee (p \to q)$$
$$\equiv (\neg p \vee \neg q) \vee (\neg p \vee q)$$
$$\equiv \neg p \vee \neg q \vee \neg p \vee q$$
$$\equiv \neg p \vee \neg p \vee \neg q \vee q$$
$$\equiv \neg p \vee \neg q \vee q$$
$$\equiv \neg p \vee T$$
$$\equiv T$$

5e. $\neg(p \to q) \to p \equiv \neg\neg(p \to q) \vee p$

$$\equiv (p \to q) \vee p$$

$$\equiv (\neg p \vee q) \vee p$$

$$\equiv \neg p \vee q \vee p$$

$$\equiv \neg p \vee p \vee q$$

$$\equiv T \vee q$$

$$\equiv T$$

f. $\neg(p \to q) \to \neg q \equiv \neg\neg(p \to q) \vee \neg q$

$$\equiv (p \to q) \vee \neg q$$

$$\equiv (\neg p \vee q) \vee \neg q$$

$$\equiv \neg p \vee (q \vee \neg q)$$

$$\equiv \neg p \vee T$$

$$\equiv T$$

6. Let $P(x)$ denote the statement "$x \leq 4$." What are these truth values?
a. $P(0)$
b. $P(4)$
c. $P(6)$

a. $P(0)$ is the statement "$0 \leq 4$", which is TRUE.
b. $P(4)$ is the statement "$4 \leq 4$", which is TRUE
c. $P(6)$ is the statement "$6 \leq 4$", which is FALSE

7. Let $Q(x, y)$ denote the statement "x is the capital of y." What are these truth values?

a. Q(Denver, Colorado)
b. Q(Detroit, Michigan)
c. Q(Massachusetts, Boston)
d. Q(New York, New York)

a. Q(Denver, Colorado) is the statement
"Denver is the capital of Colorado"
which is TRUE

b. Q(Detroit, Michigan) is the statement
"Detroit is the capital of Michigan"
which is FALSE. (Detroit is Michigan's largest city but Lansing is the capital.)

c. Q(Massachusetts, Boston) is the statement
"Massachusetts is the capital of Boston"
which is FALSE. (However, Boston is the capital of Massachusetts.)

d. Q(New York, New York) is the statement
"New York is the capital of New York"
which is FALSE.
(The city of New York is the largest city in New York state, but the capital of New York state is Albany.)

8. Translate these statements into English, where $C(x)$ is "x is a comedian" and $F(x)$ is "x is funny" and the domain consists of all people.

a. $\forall x(C(x) \to F(x))$
b. $\forall x(C(x) \land F(x))$
c. $\exists x(C(x) \to F(x))$
d. $\exists x(C(x) \land F(x))$

a. All comedians are funny

b. Everyone is a funny comedian

c. There's at least one person
who would be funny if they were a comedian

d. There's at least one funny comedian

9. Let $C(x)$ be the statement "x has a cat," let $D(x)$ be the statement "x has a dog," and let $F(x)$ be the statement "x has a ferret." Express each of these statements in terms of $C(x)$, $D(x)$, $F(x)$, quantifiers, and logical connectives. Let the domain consist of all students in your class.

a. A student in your class has a cat, a dog, and a ferret.
b. All students in your class have a cat, a dog, or a ferret.
c. Some student in your class has a cat and a ferret, but not a dog.
d. No student in your class has a cat, a dog, and a ferret.
e. For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has this animal as a pet.

a. $\exists x \left(C(x) \wedge D(x) \wedge F(x) \right)$

b. $\forall x \left(C(x) \vee D(x) \vee F(x) \right)$

c. $\exists x \left(C(x) \wedge F(x) \wedge \neg D(x) \right)$

d. $\neg \exists x \left(C(x) \wedge D(x) \wedge F(x) \right)$

e. $\left(\exists x (C(x)) \right) \wedge \left(\exists x (D(x)) \right) \wedge \left(\exists x (F(x)) \right)$

10. Let $P(x)$ be the statement "$x = x^2$." If the domain consists of the integers, what are these truth values?
a. $P(0)$
b. $P(1)$
c. $P(2)$
d. $P(-1)$
e. $\exists x P(x)$
f. $\forall x P(x)$

a. $P(0)$ is the statement "$0 = 0^2$" which is TRUE

b. $P(1)$ is the statement "$1 = 1^2$" which is TRUE

c. $P(2)$ is the statement "$2 = 2^2$" which is FALSE

d. $P(-1)$ is the statement "$-1 = (-1)^2$"
i.e. "$-1 = +1$" which is FALSE

e. TRUE because there does exist at least one x that makes $P(x)$ true. (e.g. $x = 0$ or $x = 1$)

f. FALSE because it is not the case that every x makes $P(x)$ true.
(For example, $P(x)$ isn't true if $x = 2$.)

11. Determine the truth value of each of these statements if the domain consists of all integers.
a. $\forall n(n + 1 > n)$
b. $\exists n(2n = 3n)$
c. $\exists n(n = -n)$
d. $\forall n(3n \leq 4n)$

a. True. Every integer n (positive, negative, or zero) satisfies the inequality $n+1 > n$
(i.e. $n+1$ is further right than n on the number line).

b. True. $n=0$ is an integer that satisfies $2n = 3n$.

c. True. $n=0$ is an integer that satisfies $n = -n$.

d. False. A counterexample is $n = -1$.
If $n = -1$, then $3n = -3$ and $4n = -4$ and the statement $-3 \leq -4$ is false.

12. Determine the truth value of each of these statements if the domain consists of all real numbers.

a. $\exists x(x^3 = -1)$
b. $\exists x(x^4 < x^2)$
c. $\forall x((-x)^2 = x^2$
d. $\forall x(2x > x)$

a. True. $x = -1$ is a real number that satisfies $x^3 = -1$.

b. True. $x = \frac{1}{2}$ is a real number that satisfies $x^4 < x^2$. (Note $x^4 = \frac{1}{16}$ and $x^2 = \frac{1}{4}$.)

c. True. If x is any real number, then $(-x)^2 = (-1 \cdot x)^2 = (-1)^2 \cdot x^2 = x^2$.

d. False. e.g. $x = -1$ is a counterexample. Then $2x = -2$ and $x = -1$ so $2x > x$ says $-2 > -1$, which is false.

13. Determine the truth value of each of these statements if the domain consists of all integers.

a. $\forall n(n^2 \geq 0)$
b. $\exists n(n^2 = 2)$
c. $\forall n(n^2 \geq n)$
d. $\exists n(n^2 < 0)$

a. True. The set of integers is $\{\ldots,-4,-3,-2,-1,0,1,2,3,4,\ldots\}$
The set of their squares is $\{0,1,4,9,16,\ldots\}$
It's true that $n^2 \geqslant 0$ for all integers n.

b. False. We listed all squares of integers above and there are no integers n satisfying $n^2 = 2$.

c. True. Certainly true if n is negative and also true if n is zero or positive.

n	n^2
-4	16
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9
4	16

d. False. No integer n satisfies $n^2 < 0$.