MAD2104

Suggested problems on Chapter 2 material

(Sets)

1. List the members of these sets.

a. $\{x \mid x \text{ is a real number such that } x^2 = 1\}$

b. $\{x \mid x \text{ is a positive integer less than } 12\}$

c. $\{x \mid x \text{ is the square of an integer and } x < 100\}$

d. $\{x \mid x \text{ is an integer such that } x^2 = 2\}$

a. $\{1, -1\}$

b. $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

c. $\{0, 1, 4, 9, 16, 25, 36, 49, 64, 81\}$

d. This set has no members

2. Determine whether each of these pairs of sets are equal.
a. $\{1, 3, 3, 3, 5, 5, 5, 5, 5\}$ and $\{5, 3, 1\}$
b. $\{\{1\}\}$ and $\{1, \{1\}\}$
c. $\emptyset$ and $\{\emptyset\}$

a. Yes. Both are equal to $\{1, 3, 5\}$.
(Order and repetition do not matter.)

b. No.
The first has one element, namely $\{1\}$
The second has two elements, namely 1 and $\{1\}$.

c. No.
The first has no elements.
The second has one element, namely $\emptyset$.

3. For each of the following sets, determine whether 2 is an element of that set.
a. $\{x \in \mathbb{R} \mid x \text{ is an integer greater than } 1\}$
b. $\{x \in \mathbb{R} \mid x \text{ is the square of an integer}\}$
c. $\{2, \{2\}\}$
d. $\{\{2\}, \{\{2\}\}\}$
e. $\{\{2\}, \{2, \{2\}\}\}$
f. $\{\{\{2\}\}\}$

a. Yes, because 2 is an integer greater than 1

b. No, because 2 is not the square of an integer.

c. Yes. The first of the two elements listed is the number 2.

d. No. Neither element is the number 2.
($\{2\}$ is not 2, and $\{\{2\}\}$ is not 2.)

e. No. Neither element is the number 2.
($\{2\}$ is not 2, and $\{2, \{2\}\}$ is not 2.)

f. No. The set has one element, which is $\{\{2\}\}$, which is not the number 2.

4. For each of the sets in Question 3, determine whether $\{2\}$ is an element of that set.

a. No, because $\{2\}$ is not an integer.

b. No, because $\{2\}$ is not an integer and therefore not the square of an integer.

c. Yes. The second of the two elements listed is $\{2\}$.

d. Yes. The first of the two elements listed is $\{2\}$.

e. Yes. The first of the two elements listed is $\{2\}$

f. No. The set has one element, which is $\{\{2\}\}$, which is not $\{2\}$.

5. Determine whether these statements are true or false.

a. $\emptyset \in \{\emptyset\}$
b. $\emptyset \in \{\emptyset, \{\emptyset\}\}$
c. $\{\emptyset\} \in \{\emptyset\}$
d. $\{\emptyset\} \in \{\{\emptyset\}\}$
e. $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$
f. $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$
g. $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$

a. True. $\emptyset$ is one of the elements of $\{\emptyset\}$ (the only one).

b. True. $\emptyset$ is one of the two elements of $\{\emptyset, \{\emptyset\}\}$.

c. False. $\{\emptyset\}$ is not $\emptyset$, so $\{\emptyset\}$ is not an element of $\{\emptyset\}$.

d. True. $\{\emptyset\}$ is an element of $\{\{\emptyset\}\}$ (the only one).

e. True. $\{\emptyset\}$ has one element, namely $\emptyset$.
$\{\emptyset, \{\emptyset\}\}$ has two elements, namely $\emptyset$ and $\{\emptyset\}$.
Every element of the first set is an element of the second set, and the second has an element that's not in the first.

f. True. First set has one element, $\{\emptyset\}$
Second set has two elements, $\emptyset$ and $\{\emptyset\}$

g. False. Not a <u>proper</u> subset because $\{\{\emptyset\}, \{\emptyset\}\} = \{\{\emptyset\}\}$.

6. Let $A = \{a, b, c, d\}$ and $B = \{y, z\}$. Find $A \times B$ and $B \times A$.

$$A \times B = \{ (a,y), (b,y), (c,y), (d,y), (a,z), (b,z), (c,z), (d,z) \}$$

$$B \times A = \{ (y,a), (y,b), (y,c), (y,d), (z,a), (z,b), (z,c), (z,d) \}$$

7. Translate each of these quantifications into English and determine its truth value.

a. $\forall x \in \mathbb{R}\ (x^2 \neq -1)$
b. $\exists x \in \mathbb{Z}\ (x^2 = 2)$
c. $\forall x \in \mathbb{Z}\ (x^2 > 0)$
d. $\exists x \in \mathbb{R}\ (x^2 = x)$
e. $\exists x \in \mathbb{R}\ (x^3 = -1)$
f. $\exists x \in \mathbb{Z}\ (x + 1 > x)$
g. $\forall x \in \mathbb{Z}\ (x - 1 \in \mathbb{Z})$
h. $\forall x \in \mathbb{Z}\ (x^2 \in \mathbb{Z})$

a. For all real numbers x, we have $x^2 \neq -1$. TRUE.

b. There is an integer whose square is equal to 2. FALSE.

c. For all integers x, we have $x^2 > 0$.
FALSE because 0 is an integer whose square is not GREATER than 0.

d. There is a real number that's equal to its own square.
TRUE. $x = 0$ and $x = 1$ are two examples (although we only need one)

e. There is a real number whose cube is -1. TRUE.
$x = -1$ works.

f. There is at least one integer x such that $x + 1$ is greater than x. TRUE. In fact, true for all integers.

g. For every integer, if you subtract 1 you get an integer.
TRUE.

h. For every integer, if you square it you get an integer.
TRUE.

8. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$. Find the following.

a. $A \cup B$

b. $A \cap B$

c. $A - B$

d. $B - A$

a. $\{0, 1, 2, 3, 4, 5, 6\}$

b. $\{3\}$

c. $\{1, 2, 4, 5\}$

d. $\{0, 6\}$

9. Let A, B, and C be sets. Show the following.
a. $(A \cup B) \subseteq (A \cup B \cup C)$
b. $(A \cap B \cap C) \subseteq (A \cap B)$
c. $(A - B) - C \subseteq A - C$
d. $(A - C) \cap (C - B) = \emptyset$
e. $(B - A) \cup (C - A) = (B \cup C) - A$

a. Suppose $x \in A \cup B$. We must show $x \in A \cup B \cup C$.
Since $x \in A \cup B$, that means $x \in A$ or $x \in B$ (or maybe both).
But then, by logic, the statement "$x \in A$ or $x \in B$ or $x \in C$" is true.
So $x \in A \cup B \cup C$ as required.

b. Suppose $x \in A \cap B \cap C$. We must show $x \in A \cap B$.
Since $x \in A \cap B \cap C$, that means $x \in A$ and $x \in B$ and $x \in C$.
But then in particular, we have $x \in A$ and $x \in B$.
So $x \in A \cap B$ as required.

c. Suppose $x \in (A-B)-C$. We must show $x \in A-C$.
Since $x \in (A-B)-C$, that means $x \in A-B$ and $x \notin C$.
Since $x \in A-B$, that means $x \in A$ and $x \notin B$.
In particular, it's true that $x \in A$ and $x \notin C$.
Thus $x \in A-C$ as required.

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9d. We must show that there are no x that satisfy

$x \in (A-C) \cap (C-B)$.

Suppose $x \in (A-C) \cap (C-B)$. Then $x \in A-C$ and $x \in C-B$.
Then $x \in A$ and $x \notin C$, and $x \in C$ and $x \notin B$.
In particular, we have $x \notin C$ and $x \in C$.
This cannot happen. So there is no such x.

e. PART 1. Prove $(B-A) \cup (C-A) \subseteq (B \cup C) - A$
PART 2. Prove $(B \cup C) - A \subseteq (B-A) \cup (C-A)$

Part 1. Suppose $x \in (B-A) \cup (C-A)$.
Then either $x \in B-A$ or $x \in C-A$ (or possibly both).
So either "$x \in B$ and $x \notin A$", or "$x \in C$ and $x \notin A$".
Note that $x \in B$ implies $x \in B \cup C$, and $x \in C$ implies $x \in B \cup C$.
So either way we have $x \in B \cup C$ and $x \notin A$,
so $x \in (B \cup C) - A$ as required.
Part 2. Suppose $x \in (B \cup C) - A$. Then $x \in B \cup C$ and $x \notin A$.
Then either $x \in B$ or $x \in C$ (or possibly both).
If $x \in B$ then we have "$x \in B$ and $x \notin A$", so $x \in B-A$.
If $x \in C$ then we have "$x \in C$ and $x \notin A$", so $x \in C-A$.
So we have "$x \in B-A$ or $x \in C-A$", so $x \in (B-A) \cup (C-A)$.

10. Determine whether each of these functions from $\{a, b, c, d\}$ to itself is one-to-one.

a. $f(a) = b$, $f(b) = a$, $f(c) = c$, $f(d) = d$
b. $f(a) = b$, $f(b) = b$, $f(c) = d$, $f(d) = c$
c. $f(a) = d$, $f(b) = b$, $f(c) = c$, $f(d) = d$

a. Yes, it's one-to-one, because we don't have x and y with $x \neq y$ but $f(x) = f(y)$.

b. No, it's not one-to-one, because $f(a) = f(b)$.

c. No, it's not one-to-one, because $f(a) = f(d)$.

11. Determine whether each of these functions from $\mathbb{Z}$ to $\mathbb{Z}$ is one-to-one.
a. $f(n) = n - 1$
b. $f(n) = n^2 + 1$
c. $f(n) = n^3$
d. $f(n) = \lceil n/2 \rceil$

a. The function is one-to-one.
To verify this, suppose $f(n_1) = f(n_2)$.
That means $n_1 - 1 = n_2 - 1$. Then adding 1 to both sides gives $n_1 = n_2$.
That is, $f(n_1) = f(n_2)$ implies $n_1 = n_2$, so f is one-to-one.

b. The function is not one-to-one.
For example, $f(5) = 5^2 + 1 = 25 + 1 = 26$
$f(-5) = (-5)^2 + 1 = 25 + 1 = 26$

c. The function is one-to-one.
Suppose $f(n_1) = f(n_2)$. That means $n_1^3 = n_2^3$.
Then taking cube roots tells us $n_1 = n_2$.

d. The function is not one-to-one.
For example, $f(7) = \lceil 7/2 \rceil = \lceil 3.5 \rceil = 4$

and $f(8) = \lceil 8/2 \rceil = \lceil 4 \rceil = 4$.

12. Which functions in Question 11 are onto?

a. The function is onto.
To verify this, let $m \in \mathbb{Z}$. We must find $n \in \mathbb{Z}$ that satisfies $f(n) = m$. That is, we want $n - 1 = m$. This is satisfied if we choose $n = m + 1$.

b. The function is not onto.
For example, there is no $n \in \mathbb{Z}$ satisfying $f(n) = 3$. This is because the condition $f(n) = 3$ is equivalent to $n^2 + 1 = 3$, that is, $n^2 = 2$, which has no integer solutions.

c. The function is not onto.
For example, there is no $n \in \mathbb{Z}$ satisfying $f(n) = 2$ because this is equivalent to $n^3 = 2$, which has no integer solutions.

d. The function is onto. We can illustrate using a table:

n	...	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	...
$n/2$	...	-4	-3.5	-3	-2.5	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2	2.5	...
$f(n) = \lceil n/2 \rceil$	...	-4	-3	-3	-2	-2	-1	-1	0	0	1	1	2	2	3	...

Note also that $f(2n) = \lceil 2n/2 \rceil = \lceil n \rceil = n$

13. Find the terms of the sequence $\{a_n\}$, where $a_n = 2 \cdot (-3)^n + 5^n$.

a. a_0

b. a_1

c. a_4

d. a_5

a. $a_0 = 2\cdot(-3)^0 + 5^0$

$= 2 \cdot 1 + 1 = 2 + 1 = 3$

b. $a_1 = 2\cdot(-3)^1 + 5^1$

$= 2\cdot(-3) + 5 = -6 + 5 = -1$

c. $a_4 = 2\cdot(-3)^4 + 5^4$

$= 2 \cdot 81 + 625 = 162 + 625 = 787$

d. $a_5 = 2\cdot(-3)^5 + 5^5$

$= 2\cdot(-243) + 3125 = -486 + 3125$

$= 2639$

14. Find the first five terms of the sequence defined by each of these recurrence relations and initial conditions.

a. $a_n = 6a_{n-1}$, $a_0 = 2$

b. $a_n = a_{n-1}^2$, $a_1 = 2$

c. $a_n = a_{n-1} + 3a_{n-2}$, $a_0 = 1$, $a_1 = 2$

d. $a_n = na_{n-1} + n^2 a_{n-2}$, $a_0 = 1$, $a_1 = 1$

e. $a_n = a_{n-1} + a_{n-3}$, $a_0 = 1$, $a_1 = 2$, $a_2 = 0$

a. $a_0 = 2$, $a_1 = 6a_0 = 6 \cdot 2 = 12$, $a_2 = 6a_1 = 6 \cdot 12 = 72$,
$a_3 = 6a_2 = 6 \cdot 72 = 432$, $a_4 = 6a_3 = 6 \cdot 432 = 2592$

b. $a_1 = 2$, $a_2 = a_1^2 = 2^2 = 4$, $a_3 = a_2^2 = 4^2 = 16$,
$a_4 = a_3^2 = 16^2 = 256$, $a_5 = a_4^2 = 256^2 = 65536$

c. $a_0 = 1$, $a_1 = 2$, $a_2 = a_1 + 3a_0$
$= 2 + 3 \cdot 1 = 2 + 3 = 5$

$a_3 = a_2 + 3a_1$
$= 5 + 3 \cdot 2 = 5 + 6 = 11$

$a_4 = a_3 + 3a_2$
$= 11 + 3 \cdot 5 = 11 + 15 = 26$

d. $a_0 = 1$, $a_1 = 1$, $a_2 = 2a_1 + 2^2 a_0$
$= 2 \cdot 1 + 4 \cdot 1 = 2 + 4 = 6$

$a_3 = 3a_2 + 3^2 a_1$
$= 3 \cdot 6 + 9 \cdot 1$
$= 18 + 9 = 27$

$a_4 = 4a_3 + 4^2 a_2$
$= 4 \cdot 27 + 16 \cdot 6$
$= 108 + 96 = 204$

e. on next page

14e. $a_0 = 1$

$a_1 = 2$

$a_2 = 0$

$$\begin{aligned} a_3 &= a_2 + a_0 \\ &= 0 + 1 \\ &= 1 \end{aligned}$$

$$\begin{aligned} a_4 &= a_3 + a_1 \\ &= 1 + 2 \\ &= 3 \end{aligned}$$

15. Let $a_n = 2^n + 5 \cdot 3^n$ for $n = 0, 1, 2, \ldots$.
a. Find a_0, a_1, a_2, a_3, and a_4.
b. Show that $a_2 = 5a_1 - 6a_0$, $a_3 = 5a_2 - 6a_1$, and $a_4 = 5a_3 - 6a_2$.
c. Show that $a_n = 5a_{n-1} - 6a_{n-2}$ for all integers n with $n \geq 2$.

a. $a_0 = 2^0 + 5 \cdot 3^0 = 1 + 5 \cdot 1 = 1 + 5 = 6$

$a_1 = 2^1 + 5 \cdot 3^1 = 2 + 5 \cdot 3 = 2 + 15 = 17$

$a_2 = 2^2 + 5 \cdot 3^2 = 4 + 5 \cdot 9 = 4 + 45 = 49$

$a_3 = 2^3 + 5 \cdot 3^3 = 8 + 5 \cdot 27 = 8 + 135 = 143$

$a_4 = 2^4 + 5 \cdot 3^4 = 16 + 5 \cdot 81 = 16 + 405 = 421$

b. $5a_1 - 6a_0 = 5 \cdot 17 - 6 \cdot 6 = 85 - 36 = 49 = a_2$

$5a_2 - 6a_1 = 5 \cdot 49 - 6 \cdot 17 = 245 - 102 = 143 = a_3$

$5a_3 - 6a_2 = 5 \cdot 143 - 6 \cdot 49 = 715 - 294 = 421 = a_4$

c. $5a_{n-1} - 6a_{n-2} = 5(2^{n-1} + 5 \cdot 3^{n-1}) - 6(2^{n-2} + 5 \cdot 3^{n-2})$

$= 5 \cdot 2^{n-1} + 25 \cdot 3^{n-1} - 6 \cdot 2^{n-2} - 30 \cdot 3^{n-2}$

$= 10 \cdot 2^{n-2} + 75 \cdot 3^{n-2} - 6 \cdot 2^{n-2} - 30 \cdot 3^{n-2}$

$= 4 \cdot 2^{n-2} + 45 \cdot 3^{n-2} = 2^2 \cdot 2^{n-2} + 5 \cdot 3^2 \cdot 3^{n-2}$

$= 2^n + 5 \cdot 3^n = a_n$

16. Determine whether each of these sets is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.
a. the negative integers
b. the even integers
c. the integers less than 100
d. the real numbers between 0 and $\frac{1}{2}$
e. the positive integers less than 1,000,000,000
f. the integers that are multiples of 7

a. Countably infinite.
$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ -1 & -2 & -3 & -4 & -5 \end{array} \quad \text{etc.}$$

b. Countably infinite.
$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 2 & -2 & 4 & -4 & 6 & -6 \end{array} \quad \text{etc.}$$

c. Countably infinite. ("Integers" includes negative integers.)
$$\begin{array}{ccccccccc} 1 & 2 & 3 & & 99 & 100 & 101 & 102 & 103 \\ \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 99 & 98 & 97 & & 1 & 0 & -1 & -2 & -3 \end{array} \quad \text{etc.}$$

d. Uncountable. (We can do a diagonal argument.)

e. Finite.

f. Countably infinite.
$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 7 & -7 & 14 & -14 & 21 & -21 & 28 & -28 \end{array} \quad \text{etc.}$$

17. Determine whether each of these sets is countable or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.
a. all bit strings not containing the bit 0
b. all positive rational numbers that cannot be written with denominators less than 4
c. the real numbers not containing 0 in their decimal representation
d. the real numbers containing only a finite number of 1s in their decimal representation

a. Countable.

$$\begin{aligned} 1 &\longrightarrow 1 \\ 2 &\longrightarrow 11 \\ 3 &\longrightarrow 111 \\ 4 &\longrightarrow 1111 \end{aligned}$$

etc.

b. Countable. (It's a subset of the positive rational numbers).
Can form an infinite table

$$\begin{matrix} 1/5 & 2/5 & 3/5 & 4/5 & \cdots \\ 1/6 & 2/6 & 3/6 & 4/6 & \cdots \\ 1/7 & 2/7 & 3/7 & 4/7 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{matrix}$$

and then traverse table, skipping the numbers that can be rewritten using denominators less than 4

c. Uncountable (can do variation of diagonal argument)

d. Uncountable (can do variation of diagonal argument)

18. Find **AB** if

a. $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix}$

b. $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 3 & -2 & -1 \\ 1 & 0 & 2 \end{bmatrix}$

c. $\mathbf{A} = \begin{bmatrix} 4 & -3 \\ 3 & -1 \\ 0 & -2 \\ -1 & 5 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 & 3 & 2 & -2 \\ 0 & -1 & 4 & -3 \end{bmatrix}$

a. $$AB = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2\cdot 0 + 1\cdot 1 & 2\cdot 4 + 1\cdot 3 \\ 3\cdot 0 + 2\cdot 1 & 3\cdot 4 + 2\cdot 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1 & 8+3 \\ 0+2 & 12+6 \end{bmatrix} = \begin{bmatrix} 1 & 11 \\ 2 & 18 \end{bmatrix}$$

b. $$AB = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -2 & -1 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1\cdot 3 - 1\cdot 1 & 1(-2) - 1\cdot 0 & 1(-1) - 1\cdot 2 \\ 0\cdot 3 + 1\cdot 1 & 0(-2) + 1\cdot 0 & 0(-1) + 1\cdot 2 \\ 2\cdot 3 + 3\cdot 1 & 2(-2) + 3\cdot 0 & 2(-1) + 3\cdot 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3-1 & -2-0 & -1-2 \\ 0+1 & 0+0 & 0+2 \\ 6+3 & -4+0 & -2+6 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -3 \\ 1 & 0 & 2 \\ 9 & -4 & 4 \end{bmatrix}$$

c. Similar to b, but takes longer