MAD2104

Suggested problems on Chapter 5 material (Induction and Recursion)

1. Let $P(n)$ be the statement that $1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$ for the positive integer n.
a. What is the statement $P(1)$?
b. Show that $P(1)$ is true, completing the basis step of the proof.
c. What is the inductive hypothesis?
d. What do you need to prove in the inductive step?
e. Complete the inductive step, identifying where you use the inductive hypothesis.
f. Explain why these steps show that this formula is true whenever n is a positive integer.

2a. Find a formula for

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}$$

by examining the values of this expression for small values of n.
b. Prove the formula you conjectured in part (a).

3a. Find a formula for
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$$
by examining the values of this expression for small values of n.
b. Prove the formula you conjectured in part (a).

4. Let $P(n)$ be the statement that $n! < n^n$, where n is an integer greater than 1.
a. What is the statement $P(2)$?
b. Show that $P(2)$ is true, completing the basis step of the proof.
c. What is the inductive hypothesis?
d. What do you need to prove in the inductive step?
e. Complete the inductive step.
f. Explain why these steps show that this formula is true whenever n is an integer greater than 1.

5. Let $P(n)$ be the statement that

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n},$$

where n is an integer greater than 1.
a. What is the statement $P(2)$?
b. Show that $P(2)$ is true, completing the basis step of the proof.
c. What is the inductive hypothesis?
d. What do you need to prove in the inductive step?
e. Complete the inductive step.
f. Explain why these steps show that this formula is true whenever n is an integer greater than 1.

6. Prove that $3^n < n!$ if n is an integer greater than 6.

7. Prove that $2^n > n^2$ if n is an integer greater than 4.

8. For which nonnegative integers n is $2n + 3 \leq 2^n$? Prove your answer.

9. Let $P(n)$ be the statement that a postage of n cents can be formed using just 3-cent stamps and 5-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for $n \geq 8$.
a. Show that the statements $P(8)$, $P(9)$, and $P(10)$ are true, completing the basis step of the proof.
b. What is the inductive hypothesis of the proof?
c. What do you need to prove in the inductive step?
d. Complete the inductive step for $k \geq 10$.
e. Explain why these steps show that this statement is true whenever $n \geq 8$.

10. Find $f(1)$, $f(2)$, $f(3)$, and $f(4)$ if $f(n)$ is defined recursively by $f(0) = 1$ and for $n = 0, 1, 2, \ldots$
a. $f(n+1) = f(n) + 2$.
b. $f(n+1) = 3f(n)$.
c. $f(n+1) = 2^{f(n)}$.
d. $f(n+1) = f(n)^2 + f(n) + 1$.

11. Give a recursive definition of the sequence $\{a_n\}$, $n = 1, 2, 3, \ldots$ if
a. $a_n = 6n$.
b. $a_n = 2n + 1$.
c. $a_n = 10^n$.
d. $a_n = 5$.

12. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

Show that $\mathbf{A}^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$ when n is a positive integer.