MAC2233
Suggested problems on Chapter 1 material
(functions, graphs, limits, continuity)

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1. Find the limit, if it exists.

$$\lim_{x\to 3} \frac{2x+3}{x-3}$$

Numerator approaches $2\cdot 3+3 = 6+3 = 9$

Denominator approaches $3-3 = 0$

LIMIT DOES NOT EXIST.

We could say

$$\lim_{x\to 3^+} \frac{2x+3}{x-3} = +\infty \text{ because } \frac{\text{near 9 and positive}}{\text{near 0 and positive}}$$

$$\lim_{x\to 3^-} \frac{2x+3}{x-3} = -\infty \text{ because } \frac{\text{near 9 and positive}}{\text{near 0 and negative}}$$

$$\lim_{x\to 3} \frac{2x+3}{x-3} \text{ does not exist}$$

2. Find the limit, if it exists.

$$\lim_{x \to 3} \frac{9 - x^2}{x - 3}$$

$$\lim_{x \to 3} \frac{(3+x)(3-x)}{x-3} = \lim_{x \to 3} (3+x)(-1)$$

$$= (3+3)(-1) = -6$$

3. Find the limit, if it exists.

$$\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$$

$$\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} \cdot \frac{\sqrt{x} + 2}{\sqrt{x} + 2} = \lim_{x \to 4} \frac{(\sqrt{x})^2 - 2^2}{(x-4)(\sqrt{x}+2)}$$

$$= \lim_{x \to 4} \frac{x-4}{(x-4)(\sqrt{x}+2)} = \lim_{x \to 4} \frac{1}{\sqrt{x} + 2}$$

$$= \frac{1}{\sqrt{4} + 2} = \frac{1}{2+2} = \frac{1}{4}$$

4. Find $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. If the limiting value is infinite, indicate whether it is $+\infty$ or $-\infty$.

$$f(x) = \frac{1-3x^3}{2x^3-6x+2}$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{1-3x^3}{2x^3-6x+2} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

$$= \lim_{x\to+\infty} \frac{\frac{1}{x^3} - 3}{2 - \frac{6}{x^2} + \frac{2}{x^3}} = \frac{0-3}{2-0+0} = \frac{-3}{2}$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1-3x^3}{2x^3-6x+2} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

$$= \lim_{x\to-\infty} \frac{\frac{1}{x^3} - 3}{2 - \frac{6}{x^2} + \frac{2}{x^3}} = \frac{0-3}{2-0+0} = -\frac{3}{2}$$

5. Find $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. If the limiting value is infinite, indicate whether it is $+\infty$ or $-\infty$.

$$f(x) = \frac{x^2+x-5}{1-2x-x^3}$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^2+x-5}{1-2x-x^3} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

$$= \lim_{x\to+\infty} \frac{\frac{1}{x}+\frac{1}{x^2}-\frac{5}{x^3}}{\frac{1}{x^3}-\frac{2}{x^2}-1} = \frac{0+0-0}{0-0-1} = \frac{0}{-1} = 0$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2+x-5}{1-2x-x^3} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

$$= \lim_{x\to-\infty} \frac{\frac{1}{x}+\frac{1}{x^2}-\frac{5}{x^3}}{\frac{1}{x^3}-\frac{2}{x^2}-1} = \frac{0+0-0}{0-0-1} = \frac{0}{-1} = 0$$

6. Find $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. If the limiting value is infinite, indicate whether it is $+\infty$ or $-\infty$.

$$f(x) = \frac{1-2x^3}{x+1}$$

$$f(x) = \frac{-2x^3+1}{x+1} = \frac{-2x^3 + (\text{smaller powers of } x)}{x + (\text{smaller powers of } x)}$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{-2x^3+1}{x+1} = \lim_{x\to+\infty} \frac{-2x^3}{x}$$

$$= \lim_{x\to+\infty} \underbrace{-2}_{\text{neg}}\underbrace{x^2}_{\text{pos}} = -\infty$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{-2x^3+1}{x+1} = \lim_{x\to-\infty} \frac{-2x^3}{x}$$

$$= \lim_{x\to-\infty} \underbrace{-2}_{\text{neg}}\underbrace{x^2}_{\text{pos}} = -\infty$$

7. A business manager determines that t months after production begins on a new product, the number of units produced will be P thousand, where

$$P(t) = \frac{6t^2 + 5t}{(t+1)^2}$$

What happens to production in the long run (as $t \to \infty$)?

$$P(t) = \frac{6t^2 + 5t}{t^2 + 2t + 1} \cdot \frac{\frac{1}{t^2}}{\frac{1}{t^2}} = \frac{6 + \frac{5}{t}}{1 + \frac{2}{t} + \frac{1}{t^2}}$$

If $t \to \infty$, then $P(t)$ approaches $\dfrac{6+0}{1+0+0} = 6$

So number of units produced approaches 6 thousand.

8. Find the indicated one-sided limit. If the limiting value is infinite, indicate whether it is $+\infty$ or $-\infty$.

$$\lim_{x \to 2^-} \frac{x^2 + 4}{x - 2}$$

$x \to 2^- \Rightarrow x$ slightly less than 2. $\quad x < 2$

$x - 2 < 0$

Numerator $x^2 + 4$ is near $2^2 + 4 = 4 + 4 = 8$, positive

Denominator $x - 2$ is near 0 and negative

Answer: $\displaystyle\lim_{x \to 2^-} \frac{x^2+4}{x-2} = -\infty$ because $\dfrac{\text{near 8 and pos}}{\text{near 0 and neg}}$

9. Find $\lim_{x\to 3^-} f(x)$ and $\lim_{x\to 3^+} f(x)$, where

$$f(x) = \begin{cases} 2x^2 - x & \text{if } x < 3 \\ 3 - x & \text{if } x \geq 3 \end{cases}$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (2x^2 - x)$$

because $f(x)$ is $2x^2 - x$ if x is slightly less than 3

$$= 2\cdot 3^2 - 3 = 2\cdot 9 - 3 = 18 - 3 = 15$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (3 - x)$$

because $f(x)$ is $3 - x$ if x is slightly more than 3

$$= 3 - 3 = 0$$

10. List all the values of x for which the given function is not continuous.

$$f(x) = \frac{x}{x^2 - x}$$

f is discontinuous if denominator is 0

$$x^2 - x = 0 \Rightarrow x(x-1) = 0$$

$$x = 0 \quad \text{or} \quad x - 1 = 0$$

$$x = 1$$

f is discontinuous if $x = 0$ or $x = 1$

11. List all the values of x for which the given function is not continuous.

$$f(x) = \begin{cases} 3x-2 & \text{if } x < 0 \\ x^2+x & \text{if } x \geq 0 \end{cases}$$

If $x<0$ then $f(x) = 3x-2$, which is continuous.

If $x>0$ then $f(x) = x^2+x$, which is continuous

Definition of function changes at $x=0$,
so f could be discontinuous there.
Check the one-sided limits at $x=0$.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 3x-2 = 3\cdot 0-2 = -2$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2+x = 0^2+0 = 0$$

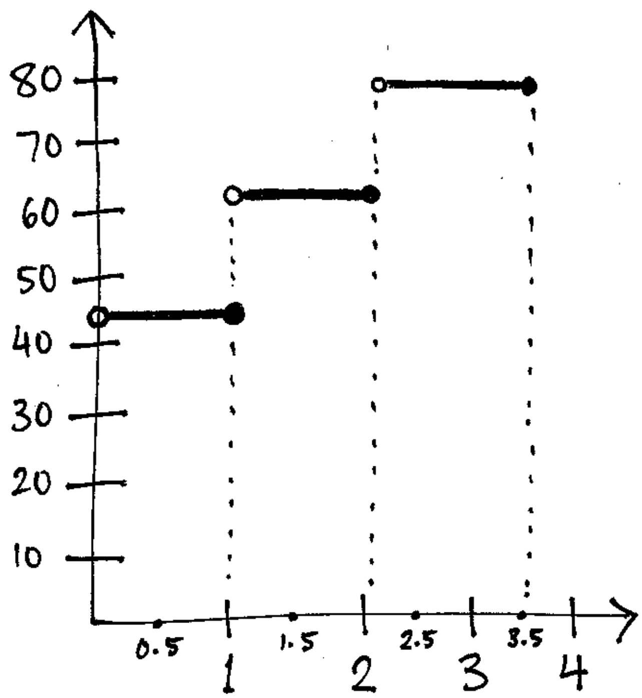
Since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$,

the function f is NOT continuous at $x=0$.

12. In 2010, the cost $p(x)$ in cents of mailing a letter weighing x ounces was

$$p(x) = \begin{cases} 44 & \text{if } 0 < x \leq 1 \\ 61 & \text{if } 1 < x \leq 2 \\ 78 & \text{if } 2 < x \leq 3.5 \end{cases}$$

Sketch the graph of $p(x)$ for $0 < x \leq 3.5$. For which of those values of x is $p(x)$ discontinuous?



$p(x)$ is discontinuous at $x=1$ and at $x=2$

According to the textbook, we would probably also say $p(x)$ is discontinuous at $x=0$ because $p(0)$ is undefined ("hole" in the graph)

13. Find the value of the constant A so that the function $f(x)$ will be continuous for all x.

$$f(x) = \begin{cases} Ax - 3 & \text{if } x < 2 \\ 3 - x + 2x^2 & \text{if } x \geq 2 \end{cases}$$

Need $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} Ax - 3 = A \cdot 2 - 3$$
$$= 2A - 3$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 3 - x + 2x^2$$
$$= 3 - 2 + 2 \cdot 2^2 = 1 + 2 \cdot 4 = 9$$

So we must have $2A - 3 = 9$

$$2A = 12$$

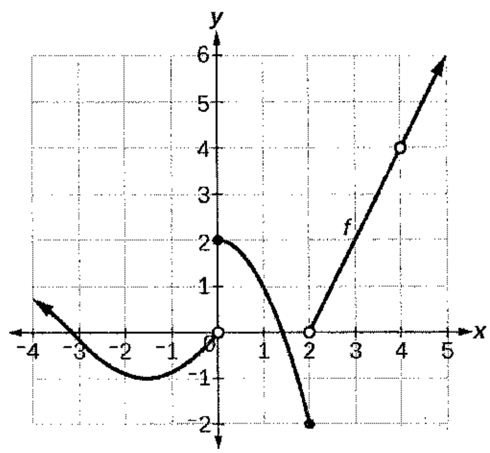
$$A = 6$$

14. Find all the indicated (one-sided and two-sided) limits, or state that they do not exist.

a. $\lim_{x\to 0^-} f(x)$ b. $\lim_{x\to 0^+} f(x)$ c. $\lim_{x\to 0} f(x)$

d. $\lim_{x\to 2^-} f(x)$ e. $\lim_{x\to 2^+} f(x)$ f. $\lim_{x\to 2} f(x)$

g. $\lim_{x\to 4^-} f(x)$ h. $\lim_{x\to 4^+} f(x)$ i. $\lim_{x\to 4} f(x)$



a. 0 b. 2 c. does not exist

d. −2 e. 0 f. does not exist

g. 4 h. 4 i. 4