MAC2233
MORE suggested problems on Chapter 2 material (differentiation)

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1. Find $\frac{dy}{dx}$ if $y = u^3 + u$ and $u = \frac{1}{\sqrt{x}}$. Simplify your answer.

$$u = \frac{1}{x^{1/2}} = x^{-1/2}$$

$$\frac{dy}{du} = 3u^2 + 1 \qquad \frac{du}{dx} = -\frac{1}{2}x^{-3/2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (3u^2+1)\cdot\left(-\frac{1}{2}\right)x^{-3/2}$$

$$= \left(3\left(\frac{1}{\sqrt{x}}\right)^2 + 1\right) \cdot \frac{-1}{2} \cdot \frac{1}{x^{3/2}}$$

$$= \left(\frac{3}{x} + 1\right) \cdot \frac{-1}{2} \cdot \frac{1}{x^{3/2}}$$

$$= \frac{3+x}{x} \cdot \frac{-1}{2} \cdot \frac{1}{x^{3/2}} = \frac{-(3+x)}{2x^{5/2}}$$

2. If $y = \dfrac{1}{u+1}$ and $u = x^3 - 2x + 5$, find the value of $\dfrac{dy}{dx}$ at $x = 0$.

$$\frac{du}{dx} = 3x^2 - 2$$

$$y = (u+1)^{-1}$$

$$\frac{dy}{du} = -1(u+1)^{-2} \cdot 1 = \frac{-1}{(u+1)^2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{-1}{(u+1)^2} \cdot (3x^2 - 2)$$

If $x = 0$ then $u = 0^3 - 2 \cdot 0 + 5 = 5$

so then $\dfrac{dy}{dx} = \dfrac{-1}{(5+1)^2} \cdot (3 \cdot 0^2 - 2)$

$$= \frac{-1}{6^2} \cdot (-2) = \frac{2}{36} = \frac{1}{18}$$

3. Differentiate the function and simplify your answer.

$$g(x) = \frac{1}{\sqrt{4x^2+1}}$$

$$g(x) = \frac{1}{(4x^2+1)^{1/2}} = (4x^2+1)^{-1/2}$$

$$g = u^{-1/2} \quad \text{and} \quad u = 4x^2+1$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\frac{dg}{du} = -\frac{1}{2}u^{-3/2} \qquad \frac{du}{dx} = 8x$$

$$\frac{dg}{dx} = \frac{dg}{du}\cdot\frac{du}{dx} = -\frac{1}{2}u^{-3/2}\cdot 8x$$

$$= -4x\cdot u^{-3/2} = \frac{-4x}{u^{3/2}} = \frac{-4x}{(4x^2+1)^{3/2}}$$

4. Differentiate the function and simplify your answer.

$$h(s) = (1 + \sqrt{3s})^5$$

$h = u^5$ and $u = 1 + \sqrt{3s}$

$u = 1 + \sqrt{3} \cdot \sqrt{s}$

$u = 1 + \sqrt{3} \cdot s^{1/2}$

$$\frac{dh}{du} = 5u^4$$

$$\frac{du}{ds} = \sqrt{3} \cdot \frac{1}{2} s^{-1/2}$$

$$= \frac{\sqrt{3}}{2} \cdot \frac{1}{s^{1/2}} = \frac{\sqrt{3}}{2\sqrt{s}}$$

$$\frac{dh}{ds} = \frac{dh}{du} \cdot \frac{du}{ds} = 5u^4 \cdot \frac{\sqrt{3}}{2\sqrt{s}} = \frac{5\sqrt{3}}{2} \cdot \frac{u^4}{\sqrt{s}}$$

$$= \frac{5\sqrt{3}}{2} \cdot \frac{(1+\sqrt{3s})^4}{\sqrt{s}} \quad \text{or} \quad \frac{5\sqrt{3}(1+\sqrt{3s})^4}{2\sqrt{s}}$$

5. Differentiate the function and simplify your answer.

$$g(x) = \sqrt{1 + \frac{1}{3x}}$$

$g = \sqrt{u}$ and $u = 1 + \frac{1}{3x} = 1 + \frac{1}{3} \cdot \frac{1}{x}$

$g = u^{1/2}$ $\qquad$ $u = 1 + \frac{1}{3} x^{-1}$

$\downarrow$ $\qquad$ $\downarrow$

$$\frac{dg}{du} = \frac{1}{2} u^{-1/2} \qquad \frac{du}{dx} = \frac{1}{3} \cdot (-1) x^{-2}$$

$$= \frac{1}{2} \cdot \frac{1}{u^{1/2}} \qquad = -\frac{1}{3} x^{-2} = \frac{-1}{3x^2}$$

$$= \frac{1}{2\sqrt{u}}$$

$$\frac{dg}{dx} = \frac{dg}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{u}} \cdot \frac{-1}{3x^2} = \frac{-1}{6x^2\sqrt{u}}$$

$$= \frac{-1}{6x^2\sqrt{1 + \frac{1}{3x}}}$$

6. Differentiate the function and simplify your answer.

$$f(x) = 2(3x+1)^4(5x-3)^2$$

$$\frac{df}{dx} = 2 \cdot \frac{d}{dx}\left((3x+1)^4 (5x-3)^2 \right)$$

$$= 2\left(\left[(3x+1)^4\right]' (5x-3)^2 + (3x+1)^4 \left[(5x-3)^2\right]' \right)$$

$$= 2\left(4(3x+1)^3 \cdot 3 \cdot (5x-3)^2 + (3x+1)^4 \cdot 2(5x-3) \cdot 5 \right)$$

$$= 2(3x+1)^3(5x-3)\left(4 \cdot 3 \cdot (5x-3) + (3x+1) \cdot 2 \cdot 5 \right)$$

$$= 4(3x+1)^3(5x-3)\left(6(5x-3) + 5(3x+1) \right)$$

$$= 4(3x+1)^3(5x-3)\left(30x - 18 + 15x + 5 \right)$$

$$= 4(3x+1)^3(5x-3)(45x - 13)$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

7. Differentiate the function and simplify your answer.

$$f(x) = \frac{(x+1)^5}{(1-x)^4}$$

$$f'(x) = \frac{[(x+1)^5]'(1-x)^4 - (x+1)^5 \cdot [(1-x)^4]'}{((1-x)^4)^2}$$

$$= \frac{5(x+1)^4 \cdot 1 \cdot (1-x)^4 - (x+1)^5 \cdot 4(1-x)^3 \cdot (-1)}{(1-x)^8}$$

$$= \frac{5(x+1)^4(1-x)^4 + 4(x+1)^5(1-x)^3}{(1-x)^8}$$

$$= \frac{(x+1)^4(1-x)^3\left(5(1-x) + 4(x+1)\right)}{(1-x)^8}$$

$$= \frac{(x+1)^4}{(1-x)^5}\left(5 - 5x + 4x + 4\right) = \frac{(x+1)^4(9-x)}{(1-x)^5}$$

8. Find the second derivative of the function.

$$y = (1 - 2x^3)^4$$

$y = u^4$ and $u = 1 - 2x^3$

$\frac{dy}{du} = 4u^3 \qquad \frac{du}{dx} = -6x^2$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 4u^3 \cdot (-6x^2) = -24u^3x^2$$

$$= -24x^2(1-2x^3)^3$$

Then $y''(x) = -24\left([x^2]'(1-2x^3)^3 + x^2\left[(1-2x^3)^3\right]'\right)$

$$= -24\left(2x(1-2x^3)^3 + x^2 \cdot 3(1-2x^3)^2 \cdot (-6x^2)\right)$$

$$= -24\left(2x(1-2x^3)^3 - 18x^4(1-2x^3)^2\right)$$

$$= -24 \cdot 2x \cdot (1-2x^3)^2 \cdot \left((1-2x^3) - 9x^3\right)$$

$$= -48x(1-2x^3)^2 \cdot (1-11x^3)$$

9. At a certain factory, the total cost of manufacturing q units is $C(q) = 0.2q^2+q+900$ dollars. It has been determined that approximately $q(t) = t^2+100t$ units are manufactured during the first t hours of a production run. Compute the rate at which the total manufacturing cost is changing with respect to time 1 hour after production begins.

Want: Rate at which cost C is changing w.r.t. t when $t=1$. That is, want $\frac{dC}{dt}$ when $t=1$.

$$C = 0.2q^2 + q + 900$$

$$\Rightarrow \frac{dC}{dq} = 0.4q + 1$$

$$q = t^2 + 100t \quad \Rightarrow \frac{dq}{dt} = 2t + 100$$

$$\frac{dC}{dt} = \frac{dC}{dq} \cdot \frac{dq}{dt} = (0.4q+1)(2t+100)$$

If $t=1$ then $q = t^2 + 100t = 1^2 + 100 \cdot 1 = 101$

So then $\frac{dC}{dt} = (0.4(101)+1)(2(1)+100)$

$$= (40.4+1)(102) = (41.4)(102)$$

$$\begin{array}{r} 41.4 \\ 102 \\ \hline 828 \\ 414 \\ \hline 4222.8 \end{array}$$

$= 4222.8$ dollars per hour

10. The number of units Q of a particular commodity that will be produced when L worker-hours of labor are employed is modeled by

$$Q(L) = 300L^{1/3}$$

Suppose that the labor level varies with time in such a way that t months from now $L(t)$ worker-hours will be employed, where

$$L(t) = \sqrt{739 + 3t - t^2}$$

for $0 \le t \le 12$.

(a) How many worker-hours will be employed in producing the commodity 5 months from now? How many units will be produced at this time?

(b) At what rate will production be changing with respect to time 5 months from now? Will production be increasing or decreasing at this time?

(a) 5 months from now, $t = 5$, and number of worker-hours is $L(5) = \sqrt{739 + 3\cdot 5 - 5^2} = \sqrt{739+15-25} = \sqrt{729} = 27$.

Number of units produced is $Q(27) = 300 \cdot 27^{1/3}$
$= 300 \cdot 3 = 900$.

(b) Want $\frac{dQ}{dt}$. $Q = 300L^{1/3} \Rightarrow \frac{dQ}{dL} = 300 \cdot \frac{1}{3} L^{-2/3} = \frac{100}{L^{2/3}}$

$L = (739 + 3t - t^2)^{1/2} \Rightarrow \frac{dL}{dt} = \frac{1}{2}(739+3t-t^2)^{-1/2} \cdot (3-2t)$

$= \frac{3-2t}{2\sqrt{739+3t-t^2}}$. If $t=5$, then $L = 27$ from part (a).

Then $\frac{dQ}{dL} = \frac{100}{27^{2/3}} = \frac{100}{9}$ and $\frac{dL}{dt} = \frac{3-10}{2\sqrt{739+15-25}} = \frac{-7}{2\cdot 27}$

So $\frac{dQ}{dt} = \frac{dQ}{dL} \cdot \frac{dL}{dt} = \frac{100}{9} \cdot \frac{-7}{2\cdot 27} = -\frac{7\cdot 50}{9\cdot 27} = \frac{-350}{243}$

which is negative, so production is decreasing at that time.

11. Suppose the total cost in dollars of manufacturing q units is $C(q) = 3q^2 + q + 500$.

(a) Use marginal analysis to estimate the cost of manufacturing the 41st unit.

(b) Compute the actual cost of manufacturing the 41st unit.

(a) $C = 3q^2 + q + 500$

$\Rightarrow C'(q) = \frac{dC}{dq} = 3 \cdot 2q + 1 + 0 = 6q + 1$

Approximate cost of manufacturing the 41^{st} unit

is $C'(40) = 6 \cdot 40 + 1 = 241$ (dollars per unit)

(b) Actual total cost of manufacturing 40 units

is $C(40) = 3 \cdot 40^2 + 40 + 500$

Actual total cost of manufacturing 41 units

is $C(41) = 3 \cdot 41^2 + 41 + 500$

Actual cost of manufacturing the 41^{st} unit

is $C(41) - C(40)$

$= (3 \cdot 41^2 + 41 + 500) - (3 \cdot 40^2 + 40 + 500)$

$= 3 \cdot 41^2 + 41 + 500 - 3 \cdot 40^2 - 40 - 500$

$= 3 \cdot 41^2 - 3 \cdot 40^2 + \underbrace{41 - 40}_{1} + \underbrace{500 - 500}_{0}$

$= 3(41^2 - 40^2) + 1 = 3(41 - 40)(41 + 40) + 1$

$= 3 \cdot 1 \cdot 81 + 1 = 243 + 1 = 244$ dollars for the 41^{st} unit

12. At a certain factory, the daily output is $Q(K) = 600K^{1/2}$ units, where K denotes the capital investment measured in units of \$1,000. The current capital investment is \$900,000. Estimate the effect that an additional capital investment of \$800 will have on the daily output.

$$Q(K) = 600K^{1/2}$$

$$\Rightarrow \frac{dQ}{dK} = Q'(K) = 600 \cdot \frac{1}{2} K^{-1/2} = 300K^{-1/2} = \frac{300}{K^{1/2}}$$

Current capital investment = \$900,000
= 900 units of \$1000, so $K = 900$.

$$Q'(900) = \frac{300}{900^{1/2}} = \frac{300}{\sqrt{900}} = \frac{300}{30} = 10.$$

That is, when $K = 900$, then $\frac{dQ}{dK} = 10$ units per thousand dollars of investment.

We can expect about another 10 units per extra thousand dollars of investment.

So, we estimate that an additional \$800 of investment will increase the daily output by about 8 units.

13. At a certain factory, the daily output is $Q(L) = 60,000L^{1/3}$ units, where L denotes the size of the labor force measured in worker-hours. Currently 1,000 worker-hours of labor are used each day. Estimate the effect on output that will be produced if the labor force is cut to 940 worker-hours.

$$Q(L) = 60,000\,L^{1/3}$$

$$\frac{dQ}{dL} = Q'(L) = 60,000 \cdot \frac{1}{3} L^{-2/3} = 20,000L^{-2/3} = \frac{20,000}{L^{2/3}}$$

Currently, $L = 1000$. So $\frac{dQ}{dL} = Q'(1000) = \frac{20,000}{1000^{2/3}}$

$$= \frac{20,000}{(1000^{1/3})^2} = \frac{20,000}{10^2} = \frac{20,000}{100} = 200 \text{ units per worker-hour.}$$

$$\frac{dQ}{dL} = 200 \text{ units per worker-hour}$$

If number of worker-hours changes from 1000 to 940 then change in L is $\Delta L = 940 - 1000 = -60$.

Then approximate change in output is

$$\Delta Q \approx \frac{dQ}{dL} \cdot \Delta L = 200 \cdot (-60) = -12,000$$

If we cut the labor force from 1000 to 940 worker-hours, we estimate that the daily output will decrease by about 12,000 units.

14. A projection made in January of 2005 determined that x years later, the average property tax on a three-bedroom home in a certain community will be $T(x) = 60x^{3/2} + 40x + 1200$ dollars. Estimate the percentage change by which the property tax will increase during the first half of the year 2013.

$$T(x) = 60x^{3/2} + 40x + 1200$$

$$\frac{dT}{dx} = T'(x) = 60 \cdot \frac{3}{2} x^{1/2} + 40 \cdot 1 + 0$$

$$= 90x^{1/2} + 40$$

In January 2013, we have $x = 8$

so $\frac{dT}{dx} = T'(8) = 90 \cdot 8^{1/2} + 40 = 90\sqrt{8} + 40$

$= 90\sqrt{4} \cdot \sqrt{2} + 40 = 180\sqrt{2} + 40$

During the first half of 2013, the change in time is $\Delta x = \frac{1}{2}$ or 0.5 years.

Approximate change in tax during that time is $\Delta T \approx \frac{dT}{dx} \cdot \Delta x = (180\sqrt{2} + 40) \cdot \frac{1}{2} = 90\sqrt{2} + 20.$

To find percentage change in tax, compare the change to the amount of tax at that time, which is

$T(8) = 60 \cdot 8^{3/2} + 40 \cdot 8 + 1200 = 60 \cdot 2^{9/2} + 1520$

$= 960\sqrt{2} + 1520$. Relative rate of change $= \frac{90\sqrt{2} + 20}{960\sqrt{2} + 1520}$

which, using a calculator, is about 0.05118 or 5.118 percent

15. It is projected that t years from now, the circulation of a local newspaper will be $C(t) = 100t^2+400t+5000$. Estimate the amount by which the circulation will increase during the next 6 months.

$$C(t) = 100t^2 + 400t + 5000$$

$$\frac{dC}{dt} = C'(t) = 100 \cdot 2t + 400 \cdot 1 + 0 = 200t + 400$$

Right now, $t=0$, so $\frac{dC}{dt} = C'(0) = 200 \cdot 0 + 400$
$= 400.$

If 6 months elapse, the change in time is
$\Delta t = 0.5$ years or $\frac{1}{2}$ year.

Approximate change in circulation during that time
is $\Delta C \approx \frac{dC}{dt} \cdot \Delta t = 400 \cdot \frac{1}{2} = 200$ copies.

We could also compute the actual change in circulation between now $(t=0)$ and six months from now $(t=0.5)$.

Actual change in circulation is $C(0.5) - C(0)$

$$= \left(100(0.5)^2 + 400(0.5) + 5000\right) - \left(100(0)^2 + 400(0) + 5000\right)$$

$$= 100 \cdot \frac{1}{4} + 400 \cdot \frac{1}{2}$$

$= 25 + 200 = 225$ copies.