MAC2233
Suggested problems on Chapter 2 material (differentiation)

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January 18, 2018

1. Compute the derivative of the given function and find the slope of the line that is tangent to its graph for the specified value of the independent variable.

$$f(x) = 2x^2 - 3x - 5, \qquad x = 0$$

$$f'(x) = 2 \cdot 2x - 3 \cdot 1 + 0$$
$$= 4x - 3$$

At $x = 0$, slope of tangent line is $4 \cdot 0 - 3 = 0 - 3 = -3$.

2. Compute the derivative of the given function and find the equation of the line that is tangent to its graph for the specified value $x = c$.

$$f(x) = \frac{-2}{x}, \qquad c = -1$$

$$f(x) = -2x^{-1}$$

$$f'(x) = -2\cdot(-1)\,x^{-2} = 2x^{-2} = \frac{2}{x^2}$$

If $x = c = -1$,

then slope of tangent line is $\frac{2}{(-1)^2} = \frac{2}{1} = 2.$

Also, if $x = -1$, then $y = f(-1) = \frac{-2}{-1} = 2.$

So the line in question goes through the point $(-1, 2)$ and has slope 2.

Equation of line: $y - 2 = 2(x - (-1))$

or $y - 2 = 2(x + 1)$

$y - 2 = 2x + 2$

$y = 2x + 4$

3. Compute the derivative of the given function and find the equation of the line that is tangent to its graph for the specified value $x = c$.

$$f(x) = 2\sqrt{x}, \qquad c = 4$$

$$f(x) = 2x^{1/2}$$

$$f'(x) = 2 \cdot \frac{1}{2} x^{-1/2} = x^{-1/2} = \frac{1}{\sqrt{x}}$$

If $x = c = 4$,

then slope of tangent line is $\frac{1}{\sqrt{4}} = \frac{1}{2}$.

Also, if $x = 4$, then $y = f(4) = 2\sqrt{4} = 2 \cdot 2 = 4$.

So the line in question goes through the point $(4,4)$ and has slope $\frac{1}{2}$.

Equation of line: $y - 4 = \frac{1}{2}(x - 4)$

or $y - 4 = \frac{x}{2} - 2$

or $y = \frac{x}{2} + 2$

4. Let $s(t) = \sqrt{t}$.

(a) Find the average rate of change of $s(t)$ with respect to t as t changes from $t = 1$ to $t = \frac{1}{4}$.

(b) Use calculus to find the instantaneous rate of change of $s(t)$ at $t = 1$, and compare with the average rate found in part (a).

(a) Average rate of change of $s(t)$ over that interval is

$$\frac{s(\frac{1}{4}) - s(1)}{\frac{1}{4} - 1} = \frac{\sqrt{\frac{1}{4}} - \sqrt{1}}{\frac{-3}{4}} = \frac{\frac{1}{2} - 1}{\frac{-3}{4}}$$

$$= \frac{-\frac{1}{2}}{\frac{-3}{4}} = -\frac{1}{2} \cdot \frac{-4}{3} = \frac{2}{3}$$

(b) Instantaneous rate of change = derivative

$$s(t) = t^{1/2} \Rightarrow s'(t) = \frac{1}{2} t^{-1/2} = \frac{1}{2\sqrt{t}}$$

At $t = 1$, the instantaneous rate of change is $s'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}$.

The answer for (b) is slightly less than the answer for (a).

5. First compute the derivative of $f(x) = x^3$ and then use it to find the slope of the tangent line to the curve $y = x^3$ at the point where $x = -1$. What is the equation of the tangent line at this point?

$$f(x) = x^3 \Rightarrow f'(x) = 3x^2$$

At $x = -1$, slope of tangent line is $f'(-1)$
$= 3(-1)^2 = 3 \cdot 1 = 3$

At $x = -1$, we have $y = f(-1) = (-1)^3 = -1$.

So, the desired tangent line has slope 3 and goes through the point $(-1, -1)$.

Equation of tangent line is

$$y - (-1) = 3(x - (-1))$$

or

$$y + 1 = 3(x + 1)$$
$$y + 1 = 3x + 3$$
$$y = 3x + 2$$

6. First compute the derivative of $f(x) = \sqrt{x}$ and then use it to:

(a) Find the equation of the tangent line to the curve $y = \sqrt{x}$ at the point where $x = 4$.

(b) Find the rate at which $y = \sqrt{x}$ is changing with respect to x when $x = 1$.

$$f(x) = x^{1/2} \Rightarrow f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

(a) At $x=4$, slope of tangent line is $f'(4)$

$$= \frac{1}{2\sqrt{4}} = \frac{1}{2\cdot 2} = \frac{1}{4}.$$

If $x=4$ then $y = f(4) = \sqrt{4} = 2$.

So, tangent line has slope $\frac{1}{4}$ and goes through $(4,2)$.

Equation of tangent line is $y - 2 = \frac{1}{4}(x-4)$

(b) Rate of change at $x=1$ is $f'(1)$

$$f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2\cdot 1} = \frac{1}{2}.$$

7. Differentiate the given function.

$$f(x) = \frac{1}{4}x^8 - \frac{1}{2}x^6 - x + 2$$

$$f'(x) = \frac{1}{4}\cdot 8x^7 - \frac{1}{2}\cdot 6x^5 - 1 + 0$$

$$= 2x^7 - 3x^5 - 1$$

8. Differentiate the given function.

$$y = \frac{3}{x} - \frac{2}{x^2} + \frac{2}{3x^3}$$

$$y = 3x^{-1} - 2x^{-2} + \frac{2}{3}x^{-3}$$

$$\frac{dy}{dx} = y' = 3\cdot(-1)x^{-2} - 2\cdot(-2)x^{-3} + \frac{2}{3}\cdot(-3)x^{-4}$$

$$= -3x^{-2} + 4x^{-3} - 2x^{-4}$$

or $$\frac{-3}{x^2} + \frac{4}{x^3} - \frac{2}{x^4}$$

9. Differentiate the given function.

$$y = -\frac{x^2}{16} + \frac{2}{x} - x^{3/2} + \frac{1}{3x^2} + \frac{x}{3}$$

$$y = -\frac{1}{16}x^2 + 2x^{-1} - x^{3/2} + \frac{1}{3}x^{-2} + \frac{1}{3}x$$

$$\frac{dy}{dx} = y' = \frac{-1}{16}\cdot 2x + 2\cdot(-1)x^{-2} - \frac{3}{2}x^{1/2}$$

$$+ \frac{1}{3}\cdot(-2)x^{-3} + \frac{1}{3}$$

$$= \frac{-x}{8} - \frac{2}{x^2} - \frac{3\sqrt{x}}{2} - \frac{2}{3x^3} + \frac{1}{3}$$

10. Differentiate the given function

$$y = \frac{x^5 - 4x^2}{x^3} \qquad [\textit{Hint: } \text{Divide first.}]$$

$$y = \frac{x^5}{x^3} - \frac{4x^2}{x^3} = x^2 - 4x^{-1}$$

$$\frac{dy}{dx} = y' = 2x - 4\cdot(-1)\,x^{-2}$$

$$= 2x + 4x^{-2}$$

$$= 2x + \frac{4}{x^2}$$

11. Differentiate the given function

$$y = x^2(x^3 - 6x + 7) \qquad [\textit{Hint:} \text{ Multiply first.}]$$

$$y = x^5 - 6x^3 + 7x^2$$

$$\frac{dy}{dx} = y' = 5x^4 - 6\cdot 3x^2 + 7\cdot 2x$$

$$= 5x^4 - 18x^2 + 14x$$

12. Find the equation of the line that is tangent to the graph of the given function at the given point.

$$y = \sqrt{x^3} - x^2 + \frac{16}{x^2}, \qquad (4, -7)$$

$$y = x^{3/2} - x^2 + 16x^{-2}$$

$$\frac{dy}{dx} = y' = \frac{3}{2}x^{1/2} - 2x + 16\cdot(-2)x^{-3}$$

$$= \frac{3}{2}\sqrt{x} - 2x - \frac{32}{x^3}$$

When $x = 4$, slope of tangent line is

$$\frac{dy}{dx} = \frac{3}{2}\sqrt{4} - 2\cdot 4 - \frac{32}{4^3} = \frac{3}{2}\cdot 2 - 8 - \frac{2^5}{2^6}$$

$$= 3 - 8 - \frac{1}{2} = -5 - \frac{1}{2} = -\frac{10}{2} - \frac{1}{2} = \frac{-11}{2}$$

So, equation of tangent line

is $y - (-7) = \frac{-11}{2}(x-4)$

or $y + 7 = -\frac{11}{2}(x-4)$ or $2y + 14 = -11(x-4)$

$2y + 14 = -11x + 44$

13. Find the equation of the line that is tangent to the graph of the given function at the given point.

$$y = (x^2 - x)(3 + 2x), \qquad (-1, 2)$$

$$y = 3x^2 + 2x^3 - 3x - 2x^2$$

$$\frac{dy}{dx} = y' = 3\cdot 2x + 2\cdot 3x^2 - 3 - 2\cdot 2x$$

$$= 6x + 6x^2 - 3 - 4x$$

$$= 6x^2 + 2x - 3$$

When $x = -1$, slope of tangent line is

$$\frac{dy}{dx} = 6(-1)^2 + 2(-1) - 3 = 6 - 2 - 3 = 1$$

So, equation of tangent line

is $y - 2 = 1(x - (-1))$

or $y - 2 = x + 1$

or $y = x + 3$

14. Find the rate of change of the given function $f(x)$ with respect to x for the prescribed value $x = c$.

$$f(x) = \frac{2}{x} - x\sqrt{x}, \qquad x = 1$$

$$f(x) = 2x^{-1} - x \cdot x^{1/2}$$

$$= 2x^{-1} - x^{3/2}$$

$$f'(x) = 2 \cdot (-1)x^{-2} - \frac{3}{2}x^{1/2}$$

$$= -\frac{2}{x^2} - \frac{3}{2}\sqrt{x}$$

If $x = 1$, then rate of change of function is $f'(1) = -\frac{2}{1^2} - \frac{3}{2}\sqrt{1}$

$$= -\frac{2}{1} - \frac{3}{2} \cdot 1 = \frac{-4}{2} - \frac{3}{2} = \frac{-7}{2}$$

15. The gross annual earnings of a certain company were $A(t) = 0.1t^2 + 10t + 20$ thousand dollars t years after its formation in 2008.

(a) At what rate were the gross annual earnings growing with respect to time in 2012?

(b) At what percentage rate were the gross annual earnings growing with respect to time in 2012?

$$A(t) = 0.1t^2 + 10t + 20$$

$$\frac{dA}{dt} = A'(t) = 0.1 \cdot 2t + 10 \cdot 1 + 0$$

$$= 0.2t + 10$$

(a) 2012 = 4 years after 2008, so $t = 4$

In 2012, rate of change of gross annual earnings is $A'(4) = 0.2(4) + 10 = 0.8 + 10 = 10.8$

10.8 thousand dollars per year

(b) Gross annual earnings in 2012 is $A(4) = 0.1 \cdot 4^2 + 10 \cdot 4 + 20$

$$= 0.1 \cdot 16 + 40 + 20 = 1.6 + 60 = 61.6$$

$$\frac{A'(4)}{A(4)} = \frac{10.8}{61.6}$$ Using calculator, this is about 17.5 %

16. Records indicate that x years after 2008, the property tax on a three-bedroom home in a certain community was $T(x) = 20x^2 + 40x + 600$ dollars.

(a) At what rate was the property tax increasing with respect to time in 2008?

(b) By how much did the tax change between 2008 and 2012?

$$T(x) = 20x^2 + 40x + 600$$

$$\frac{dT}{dx} = T'(x) = 40x + 40$$

(a) In 2008, we have $x = 0$.

At that time, the rate of change of property tax is $T'(0) = 40 \cdot 0 + 40 = 40$ (dollars per year)

(b) $2008 \Rightarrow x = 0$ and $2012 \Rightarrow x = 4$

Property tax in 2008 is $T(0) = 20 \cdot 0^2 + 40 \cdot 0 + 600 = 600$

Property tax in 2012 is $T(4) = 20 \cdot 4^2 + 40 \cdot 4 + 600$

$= 20 \cdot 16 + 160 + 600 = 320 + 760 = 1080$

Between 2008 and 2012,

the tax changed by $T(4) - T(0) = 1080 - 600 = 480$ dollars.

17. After x weeks, the number of people using a new rapid transit system was $N(x) = 6x^3 + 500x + 8000$.

(a) At what rate was the use of the system changing with respect to time after 8 weeks?

(b) By how much did the use of the system change during the eighth week?

$$N(x) = 6x^3 + 500x + 8000$$

$$\frac{dN}{dx} = N'(x) = 18x^2 + 500$$

(a) When $x=8$, the rate of change of number of users is $N'(8) = 18 \cdot 8^2 + 500 = 18 \cdot 64 + 500$
$= 1152 + 500 = 1652$ (people per week)

(b) After 7 weeks, number of users is $N(7) = 6 \cdot 7^3 + 500 \cdot 7 + 8000$
$= 6 \cdot 343 + 3500 + 8000$
$= 2058 + 3500 + 8000 = 13{,}558$

After 8 weeks, number of users is $N(8) = 6 \cdot 8^3 + 500 \cdot 8 + 8000$
$= 6 \cdot 512 + 4000 + 8000$
$= 3072 + 4000 + 8000 = 15{,}072$

Change during eighth week is $N(8) - N(7)$
$= 15{,}072 - 13{,}558 = 1514$

18. It is projected that x months from now, the population of a certain town will be $P(x) = 2x + 4x^{3/2} + 5000$.

(a) At what rate will the population be changing with respect to time 9 months from now?

(b) At what percentage rate will the population be changing with respect to time 9 months from now?

$$P(x) = 2x + 4x^{3/2} + 5000$$

$$\frac{dP}{dx} = P'(x) = 2 + 4 \cdot \frac{3}{2} x^{1/2} = 2 + 6\sqrt{x}$$

(a) 9 months from now, the rate of change of the population is $P'(9) = 2 + 6\sqrt{9} = 2 + 6 \cdot 3 = 2 + 18 = 20$ (people per month)

(b)
$$\begin{aligned} P(9) &= 2 \cdot 9 + 4 \cdot 9^{3/2} + 5000 \\ &= 18 + 4(9^{1/2})^3 + 5000 \\ &= 18 + 4 \cdot 3^3 + 5000 \\ &= 18 + 4 \cdot 27 + 5000 \\ &= 18 + 108 + 5000 = 5126 \end{aligned}$$

$$\frac{P'(9)}{P(9)} = \frac{20}{5126}$$

Using calculator, this is about 0.39 percent

0.0039 or 0.39 percent