# MAC2233
## Suggested problems on Chapter 3 material (applications of the derivative)

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1. Find the intervals of increase and decrease for the given function.

$$f(t) = t^3 + 3t^2 + 1$$

$$f'(t) = 3t^2 + 6t = 3t(t+2)$$

Crit. numbers? $f'(t)$ undefined? Never. $f'(t)=0$?
When $t=0$ or $t+2=0$, i.e. when $t=0$ or $t=-2$.

| | | $t=-2$ | | $t=0$ | |
|---|---|---|---|---|---|
| $3t$ | $- - - - - -$ | $- -$ | $- - - - -$ | $0$ | $+ + + + +$ |
| $t+2$ | $- - - - - -$ | $0$ | $+ + + + + +$ | $+ +$ | $+ + + + +$ |
| $\underbrace{3t(t+2)}_{f'(t)}$ | neg · neg = POS | | neg · pos = NEG | | pos · pos = POS |
| | $f'$ is positive<br>$f$ is INCREASING | | $f'$ is negative<br>$f$ is DECREASING | | $f'$ is positive<br>$f$ is INCREASING |

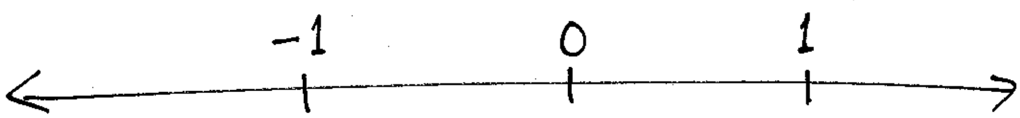
**2.** Find the intervals of increase and decrease for the given function.

$$f(x) = 3x^5 - 5x^3$$

$$\begin{aligned} f'(x) &= 3\cdot 5x^4 - 5\cdot 3x^2 \\ &= 15x^4 - 15x^2 \\ &= 15x^2(x^2 - 1) \\ &= 15x^2(x+1)(x+1) \end{aligned}$$

Crit. numbers? $f'$ never undefined. $f' = 0$ if $x = 0$ or $x+1=0$ or $x-1=0$

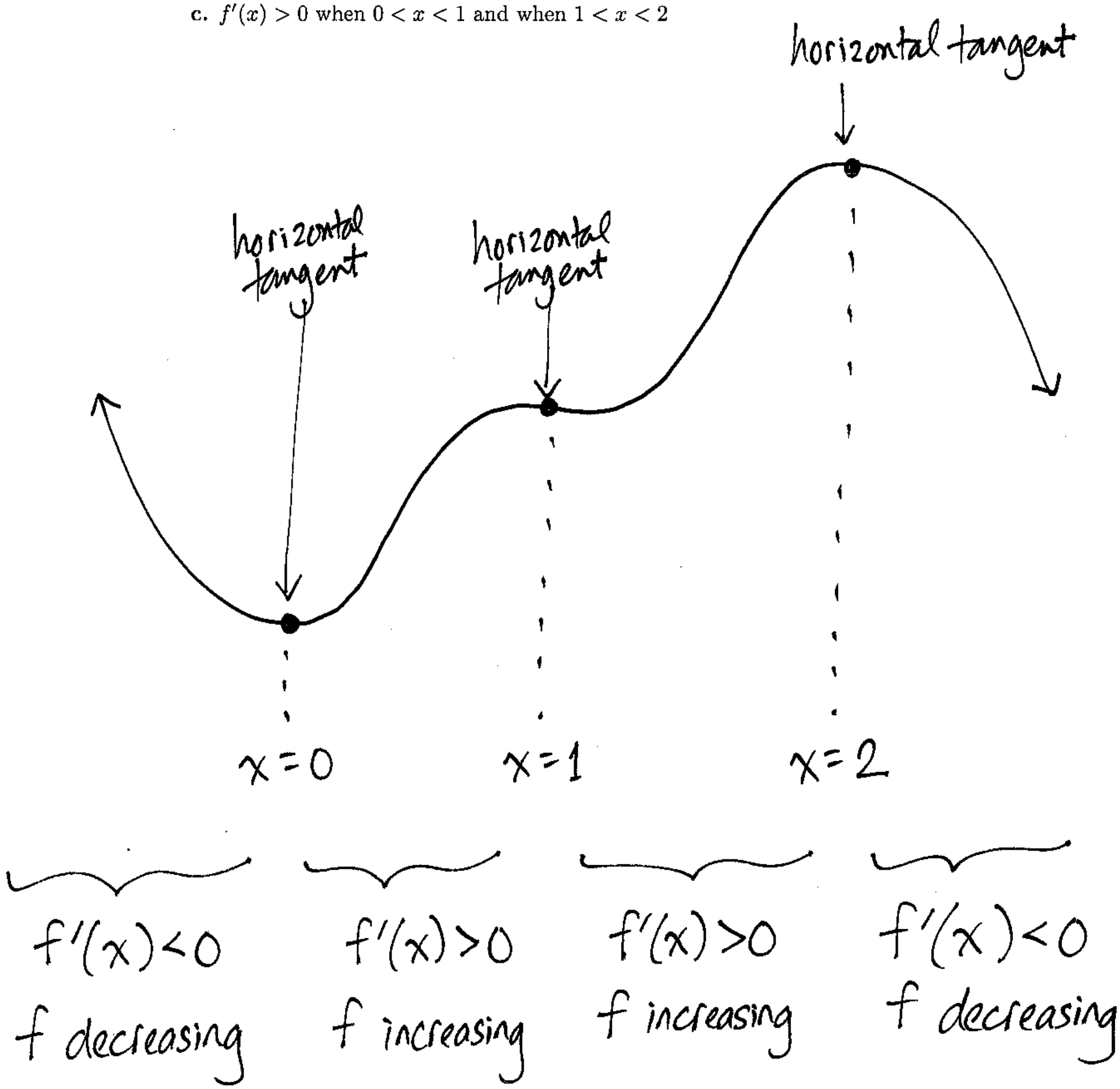
i.e. if $x = 0$ or $x = -1$ or $x = +1$.



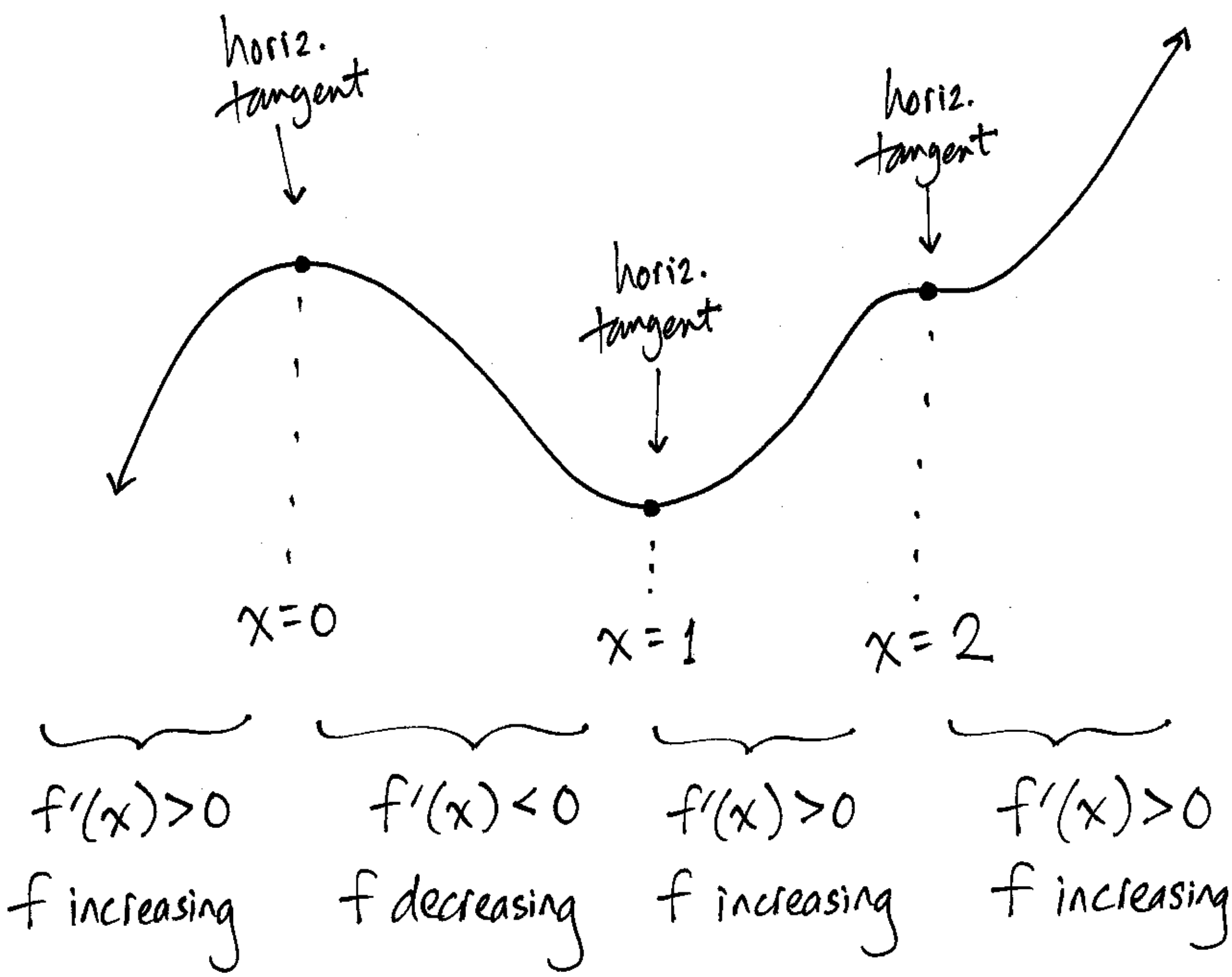

| | $(-\infty,-1)$ | $-1$ | $(-1,0)$ | $0$ | $(0,1)$ | $1$ | $(1,\infty)$ |
|---|---|---|---|---|---|---|---|
| $15x^2$ | + + + + + | + | + + + + | 0 | + + + + | + | + + + + |
| $x+1$ | − − − − − | 0 | + + + + | + | + + + + | + | + + + + |
| $x-1$ | − − − − − | − | − − − − | − | − − − − | 0 | + + + + |

| $(-\infty,-1)$ | $(-1,0)$ | $(0,1)$ | $(1,\infty)$ |
|---|---|---|---|
| $f' =$ pos·neg·neg | $f' =$ pos·pos·neg | $f' =$ pos·pos·neg | $f' =$ pos·pos·pos |
| $f'$ POS | $f'$ NEG | $f'$ NEG | $f'$ POS |
| $f$ INCREASING | $f$ DECREASING | $f$ DECREASING | $f$ INCREASING |

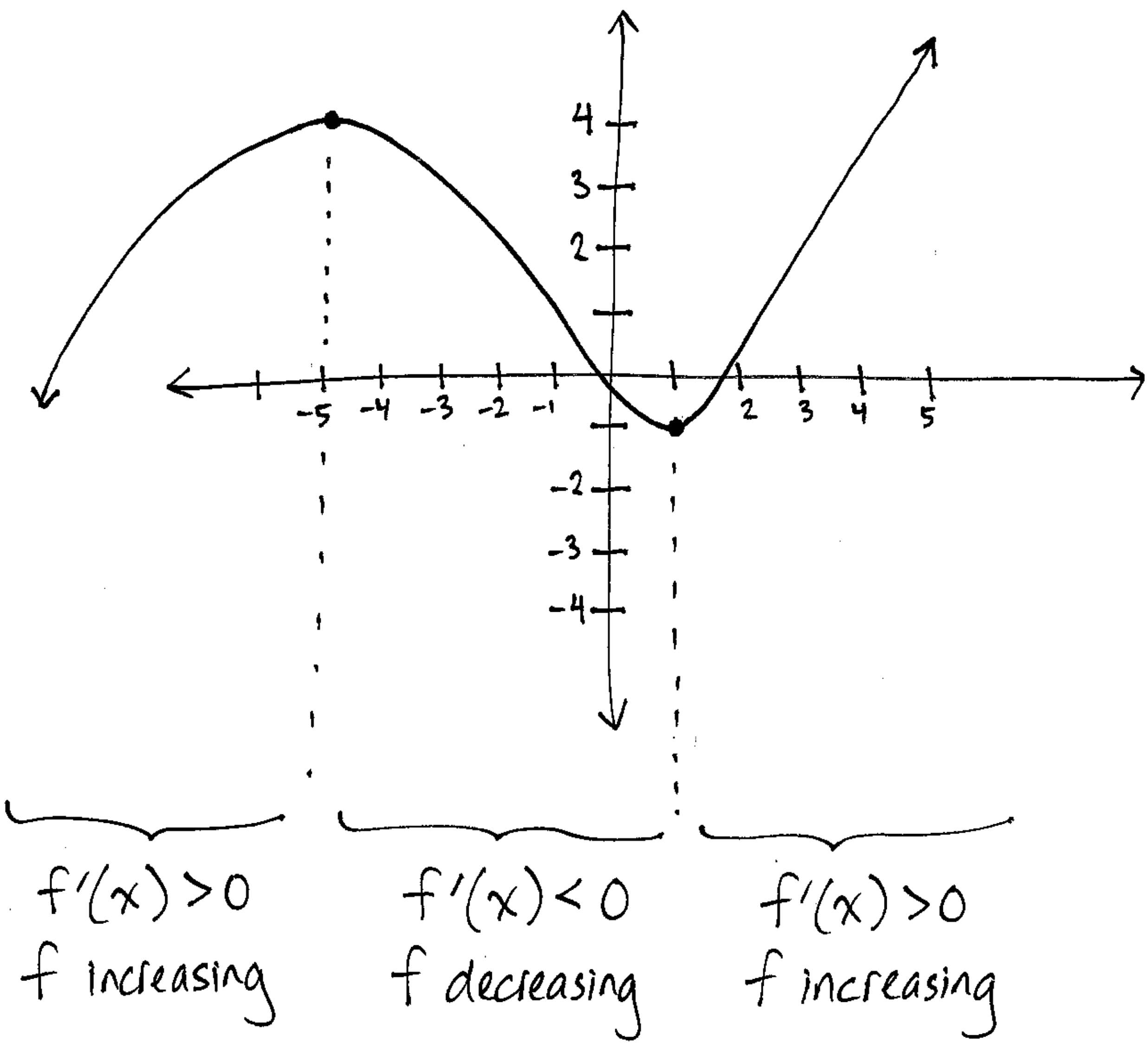
**3.** Sketch a graph of a function that has all the following properties:
**a.** $f'(0) = f'(1) = f'(2) = 0$
**b.** $f'(x) < 0$ when $x < 0$ and when $x > 2$
**c.** $f'(x) > 0$ when $0 < x < 1$ and when $1 < x < 2$



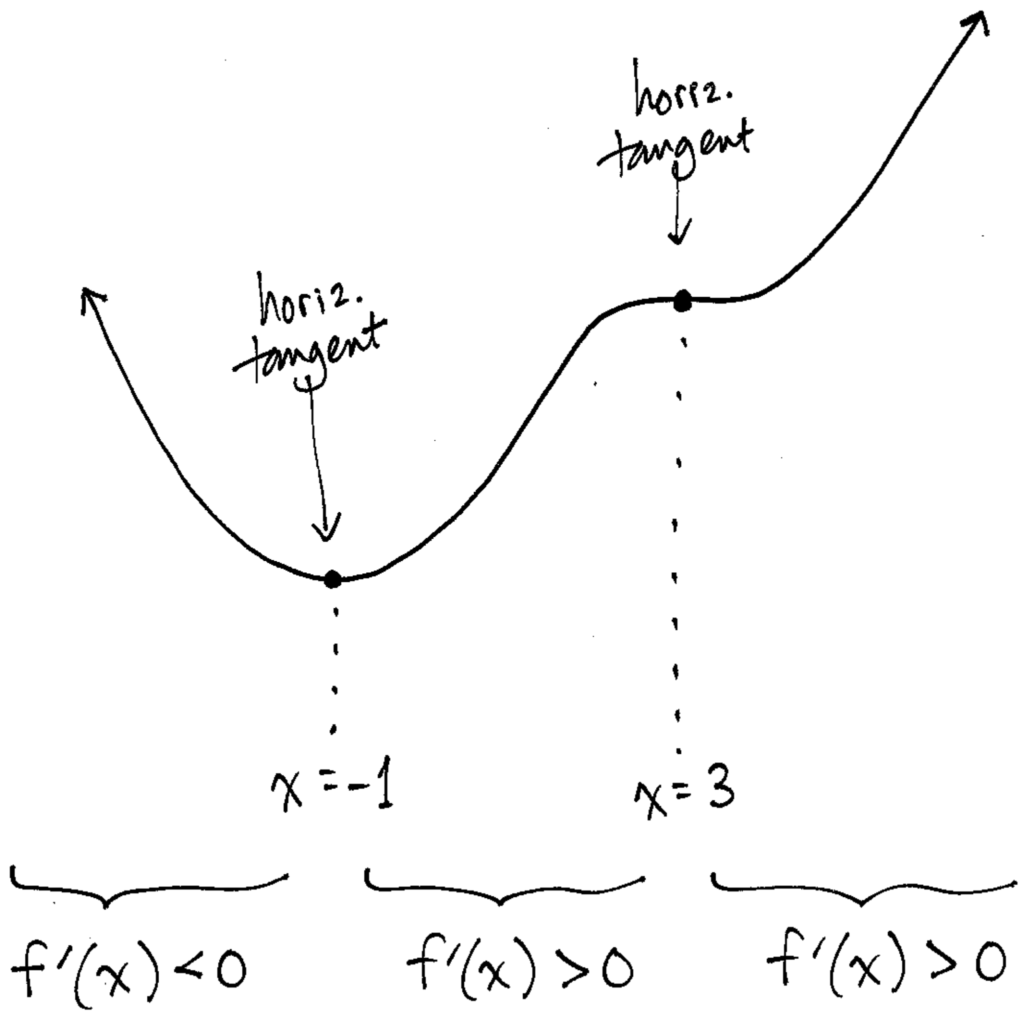
**4.** Sketch a graph of a function that has all the following properties:
**a.** $f'(0) = f'(1) = f'(2) = 0$
**b.** $f'(x) < 0$ when $0 < x < 1$
**c.** $f'(x) > 0$ when $x < 0$ and when $1 < x < 2$ and when $x > 2$



**5.** Sketch a graph of a function that has all the following properties:
**a.** $f'(x) > 0$ when $x < -5$ and when $x > 1$
**b.** $f'(x) < 0$ when $-5 < x < 1$
**c.** $f(-5) = 4$ and $f(1) = -1$



**6.** Sketch a graph of a function that has all the following properties:
**a.** $f'(x) < 0$ when $x < -1$
**b.** $f'(x) > 0$ when $-1 < x < 3$ and when $x > 3$
**c.** $f'(-1) = 0$ and $f'(3) = 0$



**7.** Determine where the function is increasing and decreasing, and where its graph is concave up and concave down. Find the relative extrema and inflection points, and sketch the graph of the function.

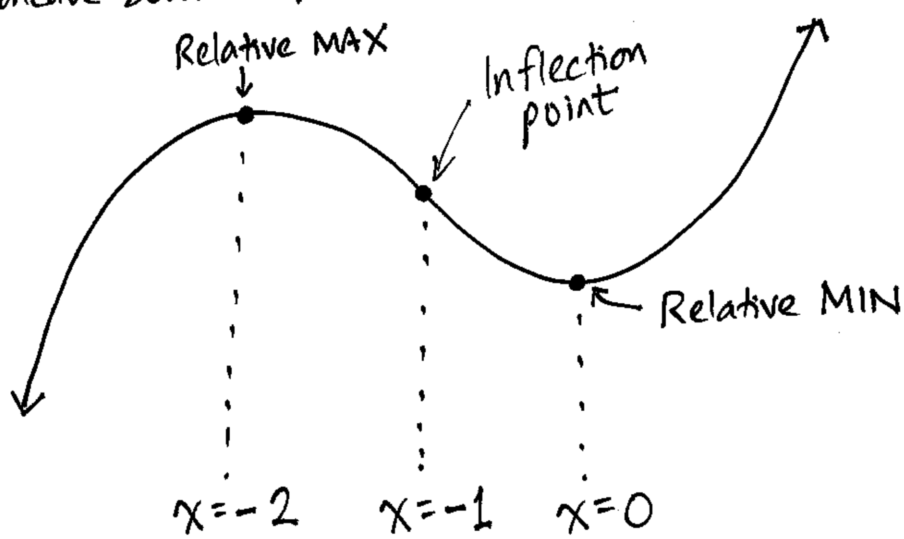
$$f(x) = x^3 + 3x^2 + 1$$

$$f'(x) = 3x^2 + 6x = 3x(x+2)$$

$$\begin{array}{lccccc} & & -2 & & 0 & \\ 3x & - - - - & - & - - - - & 0 & + + + \\ x+2 & - - - - & 0 & + + + + & + & + + + \\ & \text{f' pos} & & \text{f' neg} & & \text{f' pos} \\ & \text{f incr} \nearrow & & \text{f decr} \searrow & & \text{f incr} \nearrow \\ & & \uparrow & & \uparrow & \\ & & \text{max} & & \text{min} & \end{array}$$

$$f''(x) = 6x + 6 = 6(x+1)$$

$$\begin{array}{lccc} & & -1 & \\ x+1 & - - - - - - - & 0 & + + + + + + \\ & \text{f'' neg} & & \text{f'' pos} \\ & \text{f concave DOWN} & & \text{f concave UP} \end{array}$$



**8.** Determine where the function is increasing and decreasing, and where its graph is concave up and concave down. Find the relative extrema and inflection points, and sketch the graph of the function.

$$f(x) = x^4 - 4x^3 + 10$$

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$

0 3

$x^2$ + + + + + 0 + + + + + + + +

$x-3$ − − − − − − − − − − 0 + + + +

f′ neg, f decr ↘ | f′ neg, f decr ↘ | f′ pos, f incr ↗

↑ min

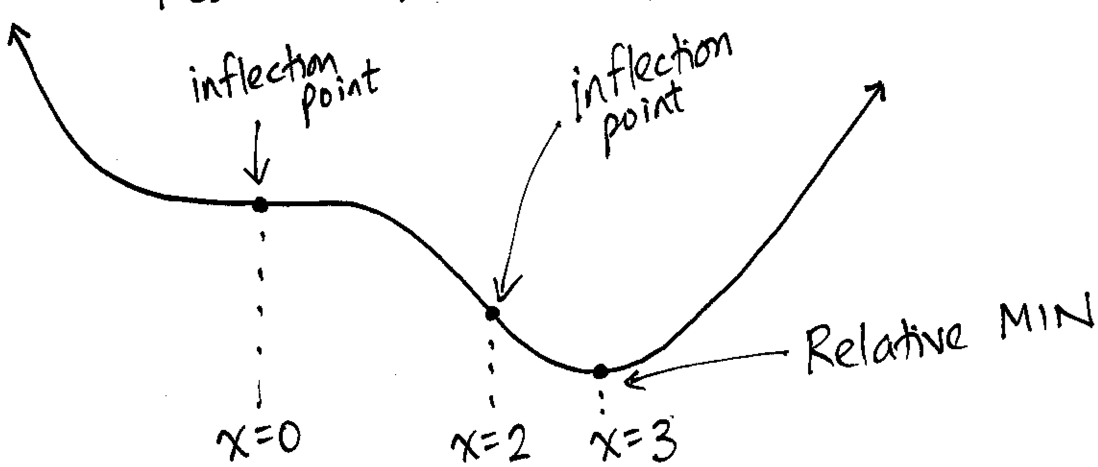
$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

0 2

$x$ − − − − 0 + + + + + + + +

$x-2$ − − − − − − − − − − 0 + + + +

f″ pos, f conc UP | f″ neg, f conc DOWN | f″ pos, f conc UP



**9.** Determine where the function is increasing and decreasing, and where its graph is concave up and concave down. Find the relative extrema and inflection points, and sketch the graph of the function.

$$f(x) = x^3 - 3x^2 + 3x + 1$$

$$f'(x) = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1)$$
$$= 3(x-1)^2$$

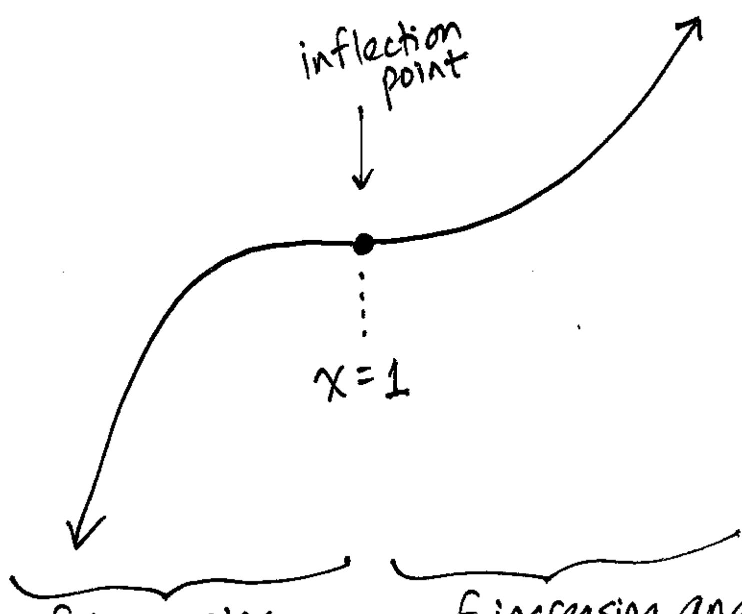
1

f′ pos — f increasing | f′ pos — f increasing

$$f''(x) = 6x - 6 = 6(x-1)$$

1

f″ neg — f concave DOWN | f″ pos — f concave UP




f increasing and concave DOWN | f increasing and concave UP

**10.** Find the relative maxima and minima of the function.

$$f(x) = 2x + 1 + \frac{18}{x}$$

$$f(x) = 2x + 1 + 18x^{-1}$$

$$f'(x) = 2 + 0 + 18\cdot(-1)x^{-2}$$

$$= 2 - 18x^{-2} = 2 - \frac{18}{x^2}$$

Critical numbers? When is $f'(x)$ undefined?
When $x=0$, but this also makes $f(x)$ undefined
so $x=0$ doesn't give us a point on the graph of $f$.
(We have a vertical asymptote there.)
When is $f'(x)=0$? $\quad 2 - \frac{18}{x^2} = 0$

$$\Rightarrow 2 = \frac{18}{x^2} \Rightarrow x^2 = 9 \Rightarrow x = 3 \text{ or } x = -3.$$

$-3 \qquad 0 \qquad 3$

| | Test value $x=-10$ | Test value $x=-1$ | Test value $x=1$ | Test value $x=10$ |
|---|---|---|---|---|
| $f'(x) = 2 - \frac{18}{x^2}$ | $f'$ pos $\left(2 - \frac{18}{100}\right)$ | $f'$ neg $\left(2 - \frac{18}{1}\right)$ | $f'$ neg $\left(2 - \frac{18}{1}\right)$ | $f'$ pos $\left(2 - \frac{18}{100}\right)$ |
| | $f$ increasing | $f$ decreasing | $f$ decreasing | $f$ increasing |

RELATIVE MAX (at $x=-3$) 

RELATIVE MIN (at $x=3$)

**11.** Find all vertical and horizontal asymptotes of the graph of the function.

$$f(x) = \frac{x}{2-x}$$

Vertical asymptotes happen when output is extreme

i.e. when $f(x)$ looks like $\frac{\text{nonzero}}{\text{zero}}$

Vertical asymptote at $x = 2$. If $x \to 2$ then $y$ is extreme

Horizontal asymptotes: check what happens when input is extreme.

$$f(x) = \frac{1x + 0}{-1x + 2} = \frac{1x + (\text{smaller powers})}{-1x + (\text{smaller powers})}$$

If $x \to \pm\infty$ then $f(x) \approx \frac{1x}{-1x} = \frac{1}{-1} = -1$.

Also, more algebraic method:

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{x}{2-x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$

$$= \lim_{x \to \pm\infty} \frac{1}{\frac{2}{x} - 1} = \frac{1}{0-1} = \frac{1}{-1} = -1.$$

Horizontal asymptote at $y = -1$. If $x$ is extreme then $y \to -1$

Overall strategy:

$f$ gives DOMAIN, INTERCEPTS, ASYMPTOTES

$f'$ gives INCREASING/DECREASING

$f''$ gives CONCAVE UP, CONCAVE DOWN

**12.** Sketch the graph of the function.

$$f(x) = \frac{x+3}{x-5}$$

Domain? $x \neq 5$. Y-intercept? $f(0) = \frac{0+3}{0-5} = \frac{3}{-5} = -\frac{3}{5}$.

X-intercept? $f(x)=0$? $\frac{x+3}{x-5} = 0 \Rightarrow x = -3$.

Asymptotes? If $x \to 5$, then $y$ is extreme. $x=5$ is vert. asymptote.

If $x$ is extreme, then $y \approx \frac{x}{x} \to 1$. $y=1$ is horiz. asymptote.

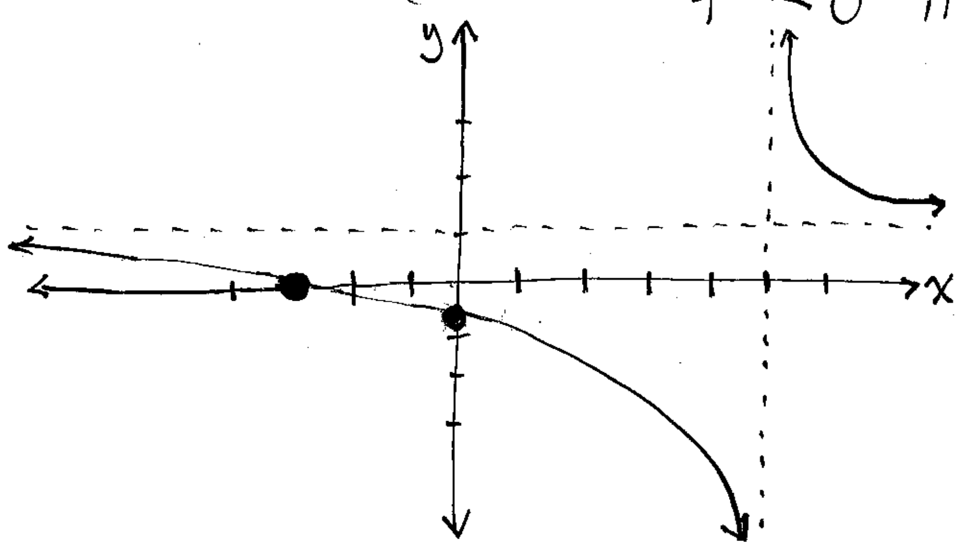
$$f'(x) = \frac{(x+3)'(x-5)-(x+3)(x-5)'}{(x-5)^2} = \frac{1(x-5)-(x+3)\cdot 1}{(x-5)^2}$$

$$= \frac{x-5-x-3}{(x-5)^2} = \frac{-8}{(x-5)^2}$$ which is always negative wherever it is defined.

$$f'(x) = -8(x-5)^{-2} \Rightarrow f''(x) = -8\cdot(-2)(x-5)^{-3}\cdot 1$$

$$= +16(x-5)^{-3} \text{ or } \frac{16}{(x-5)^3}.$$ So $f'' > 0$ if $x > 5$

$f'' < 0$ if $x < 5$



**13.** Sketch the graphs of $f(x) = x^{2/3}$ and $g(x) = (x-1)^{1/3}$.

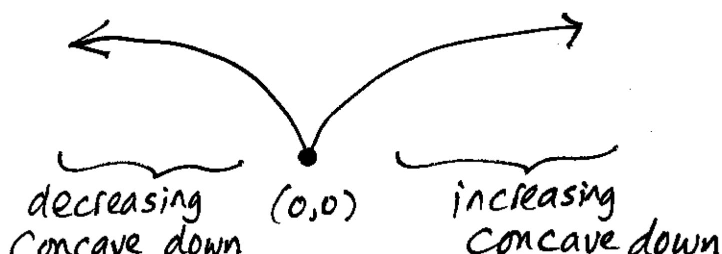
$$f(x) = x^{2/3} \Rightarrow f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$$

$f'$ is undefined when $x=0$
$f' > 0$ when $x > 0$
$f' < 0$ when $x < 0$

$$f''(x) = \frac{2}{3} \cdot \frac{-1}{3} x^{-4/3} = \frac{-2}{9} x^{-4/3} = \frac{-2}{9(x^{1/3})^4}$$

$f''$ is undefined when $x=0$ and negative otherwise

Graph of $f(x) = x^{2/3}$:




$$g(x) = (x-1)^{1/3} \Rightarrow g'(x) = \frac{1}{3}(x-1)^{-2/3} \cdot 1$$

$$= \frac{1}{3(x-1)^{2/3}} = \frac{1}{3((x-1)^{1/3})^2}$$

$g'$ is undefined if $x=1$
$g'$ is positive otherwise

$$g''(x) = \frac{1}{3} \cdot \frac{-2}{3}(x-1)^{-5/3} \cdot 1 = \frac{-2}{9}(x-1)^{-5/3}$$

$$= \frac{-2}{9(x-1)^{5/3}} = \frac{-2}{9((x-1)^{1/3})^5}$$

CONTINUED

$$g''(x) = \frac{-2}{9\left(\left((x-1)^{1/3}\right)^5\right)}$$

If $x = 1$ then $g''(x)$ is undefined

If $x > 1$ then $x-1$ is positive

$\Rightarrow (x-1)^{1/3}$ is positive

$\Rightarrow \left((x-1)^{1/3}\right)^5$ is positive

$\Rightarrow g''(x)$ is negative, so $g$ is concave down

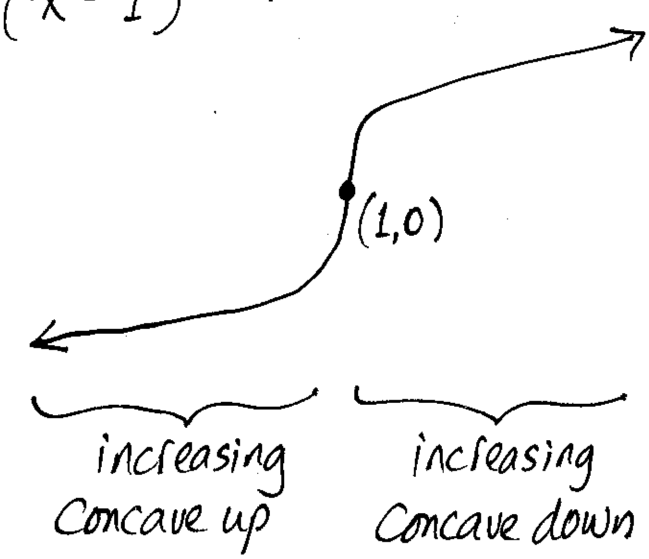
If $x < 1$ then $x-1$ is negative

$\Rightarrow (x-1)^{1/3}$ is negative

$\Rightarrow \left((x-1)^{1/3}\right)^5$ is negative

$\Rightarrow g''(x)$ is positive, so $g$ is concave up

Graph of $g(x) = (x-1)^{1/3}$:



**14.** Find the absolute maximum and absolute minimum (if any) of the given function on the specified interval.

$$f(x) = x^2 + 4x + 5, \qquad -3 \le x \le 1$$

$f'(x) = 2x + 4$

Check critical numbers and ends of domain.

$f'(x)$ undefined? Never. $f'(x) = 0$? $2x + 4 = 0$

$2x = -4$

$x = -2$

Three points to check: $x = -3$ (endpoint)

$x = -2$ (critical point)

$x = 1$ (endpoint)

$f(-3) = (-3)^2 + 4(-3) + 5$

$= 9 - 12 + 5 = 2$

$f(-2) = (-2)^2 + 4(-2) + 5$

$= 4 - 8 + 5 = 1$

$f(1) = 1^2 + 4(1) + 5$

$= 1 + 4 + 5 = 10$



The absolute max on the interval is 10 (at $x=1$)
and the absolute min on the interval is 1 (at $x=-2$)

**15.** Find the absolute maximum and absolute minimum (if any) of the given function on the specified interval.

$$f(x) = x^3 + 3x^2 + 1, \qquad -3 \le x \le 2$$

$$f'(x) = 3x^2 + 6x = 3x(x+2)$$

Crit. numbers? $f'$ is never undefined. $f'(x) = 0$?

Two critical numbers: $x = 0$, $x = -2$.

Four points to check in interval: $x=-3$, $x=-2$, $x=0$, $x=2$.

$$f(-3) = (-3)^3 + 3(-3)^2 + 1$$
$$= -27 + 3\cdot 9 + 1 = -27 + 27 + 1 = 1$$

$$f(-2) = (-2)^3 + 3(-2)^2 + 1$$
$$= -8 + 3\cdot 4 + 1 = -8 + 12 + 1 = 5$$

$$f(0) = 0^3 + 3\cdot 0^2 + 1$$
$$= 0 + 0 + 1 = 1$$

$$f(2) = 2^3 + 3\cdot 2^2 + 1$$
$$= 8 + 3\cdot 4 + 1 = 8 + 12 + 1 = 21$$

Absolute max on the interval is 21



Absolute min on the interval is 1 (which happens to occur at two different $x$ values)

**16.** Find the absolute maximum and absolute minimum (if any) of the given function on the specified interval.

$$f(x) = \frac{1}{3}x^3 - 9x + 2, \qquad 0 \le x \le 2$$

$$f'(x) = \frac{1}{3} \cdot 3x^2 - 9 \cdot 1 + 0 = x^2 - 9$$
$$= (x+3)(x-3)$$

Critical numbers: $f'$ undefined? Never.

$f'(x) = 0$? When $x = -3$ or $x = 3$.

There are no critical numbers in the interval.

We only need to check two $x$ values in the interval (the two endpoints).

$$f(0) = \frac{1}{3} \cdot 0^3 - 9 \cdot 0 + 2 = 0 - 0 + 2 = 2$$

$$f(2) = \frac{1}{3} \cdot 2^3 - 9 \cdot 2 + 2 = \frac{8}{3} - 18 + 2$$
$$= \frac{8}{3} - 16 = \frac{8}{3} - \frac{48}{3} = \frac{-40}{3}$$

The abs. max of the function on the interval is $2$

The abs. min of the function on the interval is $\frac{-40}{3}$

**17.** Find the absolute maximum and absolute minimum (if any) of the given function on the specified interval.

$$f(x) = x^5 - 5x^4 + 1, \qquad 0 \le x \le 5$$

$$f'(x) = 5x^4 - 5\cdot 4x^3 + 0$$
$$= 5x^4 - 20x^3 = 5x^3(x-4)$$

$f'$ undefined? Never. $f'(x)=0$? If $x=0$ or $x=4$.

Three points to check: $x=0$, $x=4$, $x=5$.

$$f(0) = 0^5 - 5\cdot 0^4 + 1 = 0 - 0 + 1 = 1$$
$$f(4) = 4^5 - 5\cdot 4^4 + 1$$
$$= 4\cdot 4^4 - 5\cdot 4^4 + 1$$
$$= (4-5)\cdot 4^4 + 1 = -1\cdot 4^4 + 1$$

which is an extreme negative number.

$$-4^4 + 1 = -256 + 1 = -255$$
$$f(5) = 5^5 - 5\cdot 5^4 + 1$$
$$= 5^5 - 5^5 + 1 = 1$$



The abs max on the interval is $1$

The abs min on the interval is $-255$

**18.** Find the absolute maximum and absolute minimum (if any) of the given function on the specified interval.

$$f(t) = 3t^5 - 5t^3, \qquad -2 \le t \le 0$$

$$f'(t) = 3\cdot 5t^4 - 5\cdot 3t^2$$
$$= 15t^4 - 15t^2$$
$$= 15t^2(t^2 - 1)$$
$$= 15t^2(t+1)(t-1)$$

$f'(t)$ never undefined. $f'(t) = 0$ if $t = 0$ or $t = -1$ or $t = 1$.

Only one of those critical numbers is between the endpoints of the interval. So, three points to check: $t = -2$, $t = -1$, $t = 0$.

$$f(-2) = 3(-2)^5 - 5\cdot(-2)^3$$
$$= 3\cdot(-32) - 5\cdot(-8) = -96 + 40 = -56$$

$$f(-1) = 3(-1)^5 - 5\cdot(-1)^3$$
$$= 3\cdot(-1) - 5\cdot(-1) = -3 + 5 = 2$$

$$f(0) = 3\cdot 0^5 - 5\cdot 0^3 = 0$$

The max on the interval is 2

The min on the interval is $-56$

**19.** Find the absolute maximum and absolute minimum (if any) of the given function on the specified interval.

$$f(x) = 10x^6 + 24x^5 + 15x^4 + 3, \qquad -1 \le x \le 1$$

$$f'(x) = 60x^5 + 120x^4 + 60x^3$$
$$= 60x^3(x^2 + 2x + 1)$$
$$= 60x^3(x+1)^2$$

$f'$ never undefined. $f'(x) = 0$ if $x = 0$ or $x = -1$

Three points to check: $x = -1$, $x = 0$, $x = 1$

$$f(-1) = 10(-1)^6 + 24(-1)^5 + 15(-1)^4 + 3$$
$$= 10 \cdot 1 + 24 \cdot (-1) + 15 \cdot 1 + 3$$
$$= 10 - 24 + 15 + 3 = 4$$

$$f(0) = 10 \cdot 0^6 + 24 \cdot 0^5 + 15 \cdot 0^4 + 3 = 3$$

$$f(1) = 10 \cdot 1^6 + 24 \cdot 1^5 + 15 \cdot 1^4 + 3$$
$$= 10 + 24 + 15 + 3 = 52$$

The abs max on the interval is 52

The abs min on the interval is 3

**20.** Find the absolute maximum and absolute minimum (if any) of the given function on the specified interval.

$$f(x) = (x^2 - 4)^5, \qquad -3 \le x \le 2$$

$$f'(x) = 5(x^2-4)^4 \cdot (x^2-4)' \quad \text{Chain rule}$$
$$= 5(x^2-4)^4 \cdot 2x$$
$$= 10x(x^2-4)^4$$
$$= 10x\left((x+2)(x-2)\right)^4$$

$f'$ never undefined. $f'(x)=0$ if $x=0$ or $x=-2$ or $x=2$.

Total of four points to check: $x=-3$, $x=-2$, $x=0$, $x=2$

$$f(-3) = \left((-3)^2-4\right)^5 = (9-4)^5 = 5^5$$
$$f(-2) = \left((-2)^2-4\right)^5 = (4-4)^5 = 0$$
$$f(0) = (0^2-4)^5 = (-4)^5 = -4^5$$
$$f(2) = (2^2-4)^5 = (4-4)^5 = 0$$

Abs max of function on interval is $5^5$ $(=3125)$

Abs min of function on interval is $-4^5$ $(=-1024)$

**21.** Find the absolute maximum and absolute minimum (if any) of the given function on the specified interval.

$$f(t) = \frac{t^2}{t-1}, \qquad -2 \le t \le -\frac{1}{2}$$

$$f'(t) = \frac{(t^2)'(t-1) - t^2(t-1)'}{(t-1)^2} = \frac{2t(t-1) - t^2 \cdot 1}{(t-1)^2}$$

$$= \frac{2t^2 - 2t - t^2}{(t-1)^2} = \frac{t^2 - 2t}{(t-1)^2} = \frac{t(t-2)}{(t-1)^2}$$

$f'(t)$ undefined? When $t=1$. $f'(t)=0$? When $t=0$ or $t=2$.

Three critical numbers, but no critical numbers between endpoints.

Thus, only two points to check: the two endpoints $t=-2$, $t=-\frac{1}{2}$.

$$f(-2) = \frac{(-2)^2}{-2-1} = \frac{+4}{-3} = -\frac{4}{3}$$

$$f\left(-\tfrac{1}{2}\right) = \frac{\left(-\frac{1}{2}\right)^2}{-\frac{1}{2}-1} = \frac{\frac{1}{4}}{-\frac{1}{2}-1} \cdot \frac{4}{4} = \frac{1}{-2-4} = -\frac{1}{6}$$

The absolute maximum of the function on the interval is $-\frac{1}{6}$

The absolute minimum of the function on the interval is $-\frac{4}{3}$

**22.** Find the absolute maximum and absolute minimum (if any) of the given function on the specified interval.

$$g(x) = x + \frac{1}{x}, \qquad \frac{1}{2} \le x \le 3$$

$$g(x) = x + x^{-1}$$

$$g'(x) = 1 + (-1)x^{-2} = 1 - \frac{1}{x^2} \quad \text{or} \quad \frac{x^2-1}{x^2}$$

$g'(x)$ undefined? When $x=0$. $g'(x)=0$? When $x^2=1$.

$\Rightarrow x=-1$ or $x=1$

Three critical numbers, but only one of them is between the endpoints.
Total of three points to check: $x=\frac{1}{2}$, $x=1$, $x=3$.

$$g\left(\tfrac{1}{2}\right) = \frac{1}{2} + \frac{1}{\frac{1}{2}} = \frac{1}{2} + 2 = \frac{1}{2} + \frac{4}{2} = \frac{5}{2} \quad \text{or} \quad \frac{15}{6}$$

$$g(1) = 1 + \frac{1}{1} = 1 + 1 = 2 \quad \text{or} \quad \frac{12}{6}$$

$$g(3) = 3 + \frac{1}{3} = \frac{9}{3} + \frac{1}{3} = \frac{10}{3} \quad \text{or} \quad \frac{20}{6}$$

The absolute max of the function on the interval is $\frac{10}{3} = \frac{20}{6}$



The absolute min of the function on the interval is $2 = \frac{12}{6}$

**23.** Find the absolute maximum and absolute minimum (if any) of the given function on the specified interval.

$$g(x) = \frac{1}{x^2 - 9}, \qquad 0 \le x \le 2$$

$$g(x) = (x^2 - 9)^{-1}$$

$$g'(x) = -1(x^2-9)^{-2} \cdot 2x \quad \text{(chain rule)}$$

$$g'(x) = \frac{-2x}{(x^2-9)^2}$$

$g'(x)$ undefined? When $x^2 = 9$, i.e. $x = -3$ or $x = 3$.

$g'(x) = 0$? When $x = 0$.

Three critical numbers altogether, but none between the endpoints.

Two $x$ values to check: $x = 0$ and $x = 2$.

$$g(0) = \frac{1}{0^2 - 9} = \frac{1}{-9} = \frac{-1}{9}$$

$$g(2) = \frac{1}{2^2 - 9} = \frac{1}{4-9} = \frac{1}{-5} = \frac{-1}{5}$$

The absolute max of the function on the interval is $\frac{-1}{9}$



The absolute min of the function on the interval is $\frac{-1}{5}$