24. Suppose the price at which q units of a particular commodity can be sold is

$$p(q) = 49 - q$$

and suppose the total cost of producing the q units is

$$C(q) = \frac{1}{8}q^2 + 4q + 200.$$

a. Find the revenue function $R(q)$, the profit function $P(q)$, the marginal revenue $R'(q)$, and the marginal cost $C'(q)$. Sketch the graphs of $P(q)$, $R'(q)$, and $C'(q)$ on the same coordinate axes and determine the level of production q where $P(q)$ is maximized.
b. Find the average cost $A(q) = \frac{C(q)}{q}$, and sketch the graphs of $A(q)$ and the marginal cost $C'(q)$ on the same axes. Determine the level of production q at which $A(q)$ is minimized.

(a) Revenue $R(q) = q \cdot p(q) = q(49-q) = 49q - q^2$

(# of units)$\cdot$(price per unit)

Profit $P(q) = R(q) - C(q) = (49q - q^2) - (\frac{1}{8}q^2 + 4q + 200)$

$= 49q - q^2 - \frac{1}{8}q^2 - 4q - 200 = -\frac{9}{8}q^2 + 45q - 200$

Marginal revenue $R'(q) = 49 - 2q$

Marginal cost $C'(q) = \frac{1}{8} \cdot 2q + 4 = \frac{1}{4}q + 4$

Graph on next page. Before drawing graph, what's the domain? That is, what are the possible inputs, i.e. possible q values?

q is a quantity, so we must have $q \geq 0$.

Also, price must be nonnegative, so we must have $49 - q \geq 0$ i.e. $q \leq 49$.

Domain: $0 \leq q \leq 49$.

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(a) continued. Must graph $P(q)$, $R'(q)$, and $C'(q)$.
Notice $R'(q)$ and $C'(q)$ are linear functions, easy to graph.
To graph $P(q)$, should find maximum value of $P(q)$
-and question explicitly asks us to! Find critical numbers.

$P(q) = -\frac{9}{8}q^2 + 45q - 200 \Rightarrow P'(q) = -\frac{9}{4}q + 45$

$P'(q)$ never undefined. $P'(q)=0$? $-\frac{9}{4}q + 45 = 0 \Rightarrow \frac{9}{4}q = 45$
$\Rightarrow q = 20$. Furthermore if $q<20$ then $P'(q) = -\frac{9}{4}q + 45 > 0$
if $q > 20$ then $P'(q) = -\frac{9}{4}q + 45 < 0$

So if $0<q<20$ then $P(q)$ is increasing.
if $20<q<49$ then $P(q)$ is decreasing.
If $q = 20$ then profit $P(q)$ is maximized.
For graphing purposes, let's compute some values:

$P(0) = -\frac{9}{8}\cdot 0^2 + 45\cdot 0 - 200 = -200$

$P(20) = -\frac{9}{8}\cdot 20^2 + 45\cdot 20 - 200 = -\frac{9}{8}\cdot 400 + 900 - 200$
$= -450 + 900 - 200$
$= 250$

$P(49) = -\frac{9}{8}\cdot 49^2 + 45\cdot 49 - 200 = \ldots = -696.125$

If $0 \le q \le 49$ then $R'(q) = 49 - 2q$ decreases linearly from 49 to -49
If $0 \le q \le 49$ then $C'(q) = \frac{1}{4}q + 4$ increases linearly from 4 to 16.25

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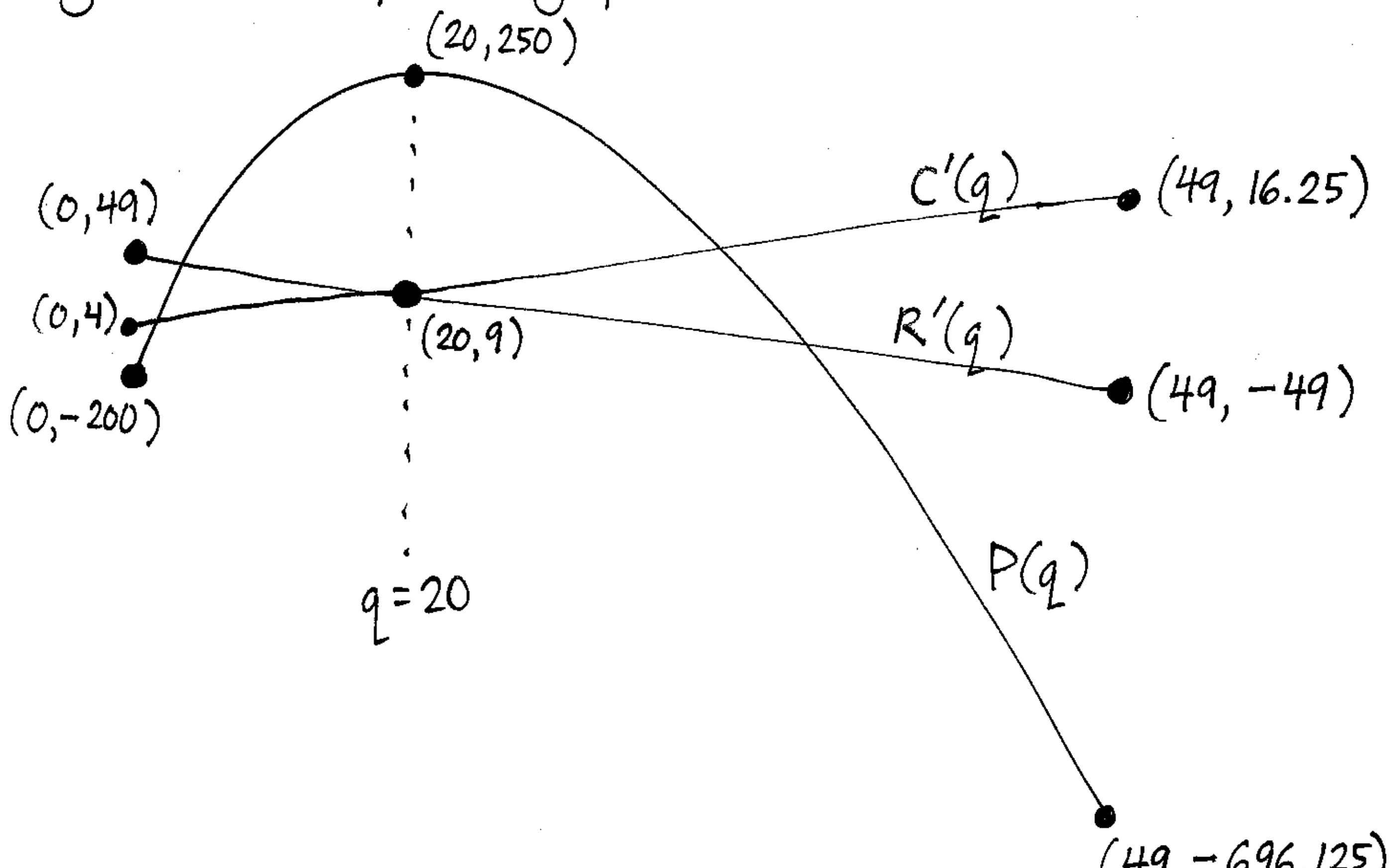
Note: Part of the point of this exercise is that the q value that maximizes $P(q)$ will also satisfy $R'(q) = C'(q)$. This is because maximum of P occurs at critical point for P i.e. $P'(q)$, so $R'(q) - C'(q) = 0$.

Double-check: $R'(20) = 49 - 2 \cdot 20 = 49 - 40 = 9$

$C'(20) = \frac{1}{4} \cdot 20 + 4 = 5 + 4 = 9$ ✓

Rough overall shape of graph:



PART B IS ON NEXT PAGE

24b. $C(q) = \frac{1}{8}q^2 + 4q + 200$

$$A(q) = \frac{C(q)}{q} = \frac{1}{8}q + 4 + \frac{200}{q} = \frac{1}{8}q + 4 + 200q^{-1}$$

Want to minimize $A(q)$ and also graph $A(q)$.

$$A'(q) = \frac{1}{8} \cdot 1 + 0 + 200 \cdot (-1)q^{-2} = \frac{1}{8} - \frac{200}{q^2}.$$

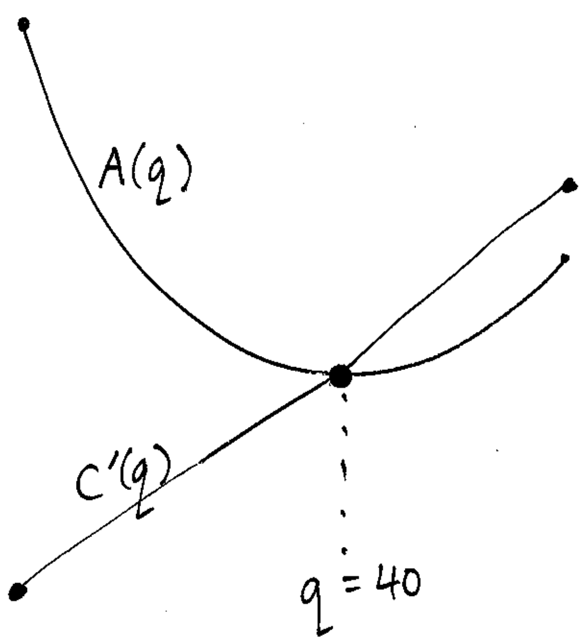
$A'(q) = 0$? $\quad \frac{1}{8} - \frac{200}{q^2} \Rightarrow \frac{1}{8} = \frac{200}{q^2} \Rightarrow q^2 = 1600$

$\Rightarrow q = 40$

Now note that if $q < 40$ (e.g. $q = 1$) then $A'(q) = \frac{1}{8} - \frac{200}{q^2}$ is neg

If $q > 40$ then $A'(q) = \frac{1}{8} - \frac{200}{q^2}$ is positive

Rough overall shape of graph:



25. Suppose the demand q and price p for a certain commodity are related by the linear equation $q = 240 - 2p$ (for $0 \leq p \leq 120$).
a. Express the elasticity of demand as a function of p.
b. Calculate the elasticity of demand when the price is $p = 100$. Interpret your answer.
c. Calculate the elasticity of demand when the price is $p = 50$. Interpret your answer.
d. At what price is the elasticity of demand equal to 1? What is the economic significance of this price?

(a) $E = -\frac{p}{q} \cdot \frac{dq}{dp}$. If $q = 240 - 2p$ then $\frac{dq}{dp} = 0 - 2 \cdot 1 = -2$.

So $E = \frac{-p}{240-2p} \cdot (-2) = \frac{2p}{240-2p}$.

(b) If $p = 100$ then $E = \frac{2 \cdot 100}{240 - 2 \cdot 100} = \frac{200}{240-200} = \frac{200}{40} = 5$

This means demand is elastic, and a 1% increase in price will cause a 5% decrease in demand.

(c) If $p = 50$ then $E = \frac{2 \cdot 50}{240 - 2 \cdot 50} = \frac{100}{240-100} = \frac{100}{140} = \frac{5}{7}$ or about 0.71

This means demand is inelastic, and a 1% increase in price will cause a 0.71 % decrease in demand.

(d) $E = 1 \Rightarrow \frac{2p}{240-2p} = 1 \Rightarrow 2p = 240 - 2p \Rightarrow 4p = 240$

$\Rightarrow p = 60$. This means 60 is the price that maximizes total revenue.

26. Compute the elasticity of demand for the given demand function $D(p)$ and determine whether the demand is elastic, inelastic, or of unit elasticity at the indicated price p.

$$D(p) = 200 - p^2, \qquad p = 10$$

Note: This time, they're calling demand D rather than q, but demand is demand. $q = 200 - p^2$

$$E = -\frac{p}{q} \cdot \frac{dq}{dp} \qquad \Rightarrow \frac{dq}{dp} = -2p$$

$$= \frac{-p}{200 - p^2} \cdot (-2p) = \frac{2p^2}{200 - p^2}$$

If $p = 10$ then $E = \dfrac{2 \cdot 10^2}{200 - 10^2} = \dfrac{200}{200 - 100}$

$$= \frac{200}{100} = 2.$$

Demand is <u>elastic</u> if $p = 10$.

(At that price, a 1% increase in price will cause a 2% decrease in demand.)

27. An art gallery offers 50 prints by a famous artist. If each print in the limited edition is priced at p dollars, it is expected that $q = 500 - 2p$ prints will be sold.

a. What limitations are there on the possible range of the price p?

b. Find the elasticity of demand. Determine the values of p for which the demand is elastic, inelastic, and of unit elasticity.

c. Interpret the results of part (b) in terms of the behavior of the total revenue as a function of unit price p.

d. If you were the owner of the gallery, what price would you charge for each print? Explain the reasoning behind your decision.

(a) If there are only 50 prints, then we should only consider demanded quantity q between 0 and 50.

$$0 \le q \le 50$$

$$0 \le 500 - 2p \le 50$$

$$0 \le 500 - 2p \qquad\qquad 500 - 2p \le 50$$

$$2p \le 500 \qquad\qquad 500 - 50 \le 2p$$

$$p \le 250 \qquad\qquad 450 \le 2p$$

$$225 \le p$$

We should only consider values of p between 225 and 250.

(Note: according to the model $q = 500 - 2p$, what would happen if we charged less than 225 dollars? Say $p = 210$. Then the public want to buy $q = 500 - 2 \cdot 210 = 500 - 420 = 80$ prints, but we only have 50 to sell. We would have sold all 50 if we charged 225, so there's no point in charging less.)

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(b) $E = -\frac{p}{q} \cdot \frac{dq}{dp}$

$q = 500 - 2p \Rightarrow \frac{dq}{dp} = -2$

$E = \frac{-p}{500-2p} \cdot (-2) = \frac{2p}{500-2p}$

$E > 1$? $\quad E < 1$? $\quad E = 1$?

$\frac{2p}{500-2p} > 1 \quad \frac{2p}{500-2p} < 1 \quad \frac{2p}{500-2p} = 1$

$2p > 500-2p \quad 2p < 500-2p \quad 2p = 500-2p$

$4p > 500 \quad 4p < 500 \quad 4p = 500$

$p > 125 \quad p < 125 \quad p = 125$

If $p > 125$ then demand is elastic
If $p < 125$ then demand is inelastic
If $p = 125$ then demand has unit elasticity.

(c) If we charge more than 125 dollars per print then demand will be very sensitive to changes in price and an increase in price will make us lose revenue.

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(c) continued

If we charge less than 125 dollars per print then demand will not be very sensitive to changes in price and an increase in price will make us gain revenue.

(d) Based on all information given, we should charge 225 dollars per print.

(If we had more prints to sell, we could maximize revenue by charging 125 dollars per print, but we can't sell more than 50 prints.)

We can also illustrate what's happening with a table:

price p	quantity $q=500-2p$	Revenue = price · quantity
225	$500-450=50$	$225\cdot 50 = 11{,}250$
230	$500-460=40$	$230\cdot 40 = 9{,}200$
235	$500-470=30$	$235\cdot 30 = 7{,}050$
240	$500-480=20$	$240\cdot 20 = 4{,}800$
245	$500-490=10$	$245\cdot 10 = 2{,}450$
250	$500-500=0$	$250\cdot 0 = 0$

If we had more prints, we could sell 60 prints for \$220 each, or 70 prints for \$215 each, and so on. But we don't.

28. A bookstore can obtain a certain novel from the publisher at a cost of \$3 per book. The bookstore has been offering the novel at a price of \$15 per copy and, at this price, has been selling 200 copies a month. The bookstore is planning to lower its price to stimulate sales and estimates that for each \$1 reduction in the price, 20 more books will be sold each month. At what price should the bookstore sell the novel to generate the greatest possible profit?

Current price per unit: 15. Current demanded quantity: 200.

If we reduce price by 1 dollar, demanded quantity increases by 20.

If we reduce price by x dollars, demanded quantity increases by $20x$.

So: if price per unit is $15 - x$

then demanded quantity is $200 + 20x$.

Cost = 3 dollars per unit

Total cost = (3 dollars per unit)$\cdot$($200 + 20x$ units)

$= 600 + 60x$

Total revenue = (price per unit)$\cdot$(quantity sold)

$$= (15 - x)(200 + 20x)$$
$$= 3000 + 300x - 200x - 20x^2$$
$$= 3000 + 100x - 20x^2$$

Profit = $P = P(x)$ = revenue − cost

$$= (3000 + 100x - 20x^2) - (600 + 60x)$$
$$= 2400 + 40x - 20x^2$$

$$\frac{dP}{dx} = P'(x) = 40 - 40x$$

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Want to maximize $P(x)$. Critical numbers?

$P'(x)$ is never undefined, and $P'(x) = 0$ if $x = 1$.

Furthermore if $x < 1$, say $x = \frac{1}{2}$, then $P'(x) = 40 - 4x$ is positive

if $x > 1$, say $x = 2$, then $P'(x) = 40 - 4x$ is negative

Therefore $P(x)$ really is maximized when $x = 1$.

Price per unit is $15 - x$

so, to maximize profit, we should charge $15 - 1 = 14$ dollars.

We can double-check whether this seems right by making a table:

Price	Quantity	Cost	Revenue	Profit
15	200	$3 \cdot 200 = 600$	$15 \cdot 200 = 3000$	$3000 - 600 = 2400$
14	220	$3 \cdot 220 = 660$	$14 \cdot 220 = 3080$	$3080 - 660 = 2420$
13	240	$3 \cdot 240 = 720$	$13 \cdot 240 = 3120$	$3120 - 720 = 2400$
12	260	$3 \cdot 260 = 780$	$12 \cdot 260 = 3120$	$3120 - 780 = 2340$
11	280	$3 \cdot 280 = 840$	$11 \cdot 280 = 3080$	$3080 - 840 = 2240$