MAC2233

Suggested problems on Chapter 4 material (exponential and logarithmic functions)

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March 20, 2018

1. Suppose \$1,000 is invested at an annual interest rate of 7%. Compute the future value of the investment after 10 years if the interest is compounded:
a. Annually
b. Quarterly
c. Monthly
d. Continuously

a. Future value $= 1000 \cdot \left(1 + \frac{.07}{1}\right)^{10} = 1000 \cdot (1.07)^{10}$
$= 1967.15$

b. Future value $= 1000 \cdot \left(1 + \frac{.07}{4}\right)^{40} = 1000 \cdot (1.0175)^{40}$
$= 2001.60$

c. Future value $= 1000 \cdot \left(1 + \frac{.07}{12}\right)^{120} = 2009.66$

d. Future value $= 1000 \cdot e^{.07 \times 10} = 1000\, e^{.7}$
$= 2013.75$

2. Suppose \$5,000 is invested at an annual interest rate of 10%. Compute the future value of the investment after 10 years if the interest is compounded:

a. Annually
b. Semiannually
c. Daily (using 365 days per year)
d. Continuously

$P = 5000 \quad r = 0.10 = 0.1 \quad t = 10$

If interest is compounded k times per year then future value $= P \cdot \left(1 + \frac{r}{k}\right)^{kt}$

If interest is compounded continuously then future value $= P \cdot e^{rt}$

a. Future value $(k=1)$ $= 5000 \cdot \left(1 + \frac{0.1}{1}\right)^{10} = 5000 \cdot (1.1)^{10}$
$= 12,968.71$

b. Future value $(k=2)$ $= 5000 \cdot \left(1 + \frac{0.1}{2}\right)^{20} = 5000 \cdot (1.05)^{20}$
$= 13,266.49$

c. Future value $(k=365)$ $= 5000 \cdot \left(1 + \frac{0.1}{365}\right)^{3650} = 13,589.55$

d. Future value $= 5000 \cdot e^{0.1 \times 10} = 5000e$
$= 13,591.41$

3. Find the present value of \$10,000 over a term of 5 years at an annual interest rate of 7% if interest is compounded:

a. Annually
b. Quarterly
c. Daily (using 365 days per year)
d. Continuously

We want to find present value. \$10,000 is the future value.

$B = 10{,}000 \qquad t = 5 \qquad r = 0.07$

If interest is compounded k times per year

then present value $= B \cdot \left(1 + \frac{r}{k}\right)^{-kt}$

If interest is compounded continuously

then present value $= B \cdot e^{-rt}$

a. $(k=1)$ Present value $= 10{,}000 \cdot \left(1 + \frac{.07}{1}\right)^{-5} = 10{,}000 \cdot (1.07)^{-5}$

$= 7{,}129.86$

b. $(k=4)$ Present value $= 10{,}000 \cdot \left(1 + \frac{.07}{4}\right)^{-20} = 10{,}000 \cdot (1.0175)^{-20}$

$= 7{,}068.25$

c. $(k=365)$ Present value $= 10{,}000 \cdot \left(1 + \frac{.07}{365}\right)^{-365 \times 5}$

$= 7{,}047.12$

d. Present value $= 10{,}000 \cdot e^{-0.07 \times 5}$

$= 10{,}000 \cdot e^{-0.35} = 7{,}046.88$

4. Find the present value of $25,000 over a term of 10 years at an annual interest rate of 5% if interest is compounded:

a. Semiannually

b. Monthly

c. Continuously

$B = 25{,}000 \quad t = 10 \quad r = 0.05$

a. $k = 2$. Present value $= 25{,}000 \cdot \left(1 + \frac{0.05}{2}\right)^{-20}$

$= 25{,}000 \cdot (1.025)^{-20}$

$= 15{,}256.77$

b. $k = 12$. Present value $= 25{,}000 \cdot \left(1 + \frac{0.05}{12}\right)^{-120}$

$= 15{,}179.03$

c. Present value $= 25{,}000 \cdot e^{-0.05 \times 10}$

$= 25{,}000 \cdot e^{-0.5}$

$= 15{,}163.27$

5. Find the effective interest rate r_e for an annual interest rate of 6% that is compounded quarterly.

$$1 + r_e = \left(1 + \frac{r}{k}\right)^k$$

$$1 + r_e = \left(1 + \frac{0.06}{4}\right)^4 = (1.015)^4$$

$$r_e = (1.015)^4 - 1 = 1.06136 - 1$$

$$= 0.06136 \quad \text{or} \quad 6.136\%$$

6. Find the effective interest rate r_e for an annual interest rate of 8% that is compounded daily (use $k = 365$).

$$1 + r_e = \left(1 + \frac{r}{k}\right)^k$$

$$1 + r_e = \left(1 + \frac{0.08}{365}\right)^{365}$$

$$1 + r_e = 1.08328$$

$$r_e = 0.08328$$

$$\text{or} \quad 8.328\%$$

7. Esmeralda needs \$5,000 for a trip to Peru when she graduates from college in 4 years. How much must she invest now at an annual interest rate of 5% compounded continuously to achieve her goal?

Future value $B = 5000$. Want to know present value P.

$B = P \cdot e^{rt}$ or $P = B \cdot e^{-rt}$. $t = 4$, $r = 0.05$

$$P = 5000 \cdot e^{-0.05 \times 4}$$

$$= 5000 \cdot e^{-0.2} = 4093.65$$

is the amount she should invest today

8. Lyle buys a rare stamp for \$500. If the annual rate of inflation is 4%, how much should he ask when he sells it in 5 years to break even?

In other words, what's the future value of \$500 today if $r = 0.04$ and $t = 5$ years.

We are probably supposed to assume that inflation is happening continuously.

$$B = P \cdot e^{rt}$$

$$= 500 \cdot e^{0.04 \times 5} = 500 \cdot e^{0.2}$$

$$= 610.70$$

9. How quickly will money double if it is invested at an annual interest rate of 7% compounded continuously?

Continuous compounding. How long does it take money to double?

Want $P \cdot e^{rt} = 2 \cdot P$ or equivalently $e^{rt} = 2$.

Here $r = 0.07$. So $e^{0.07t} = 2$

$$\ln\left(e^{0.07t}\right) = \ln 2$$

$$0.07t = \ln 2$$

$$t = \frac{\ln 2}{0.07} = 9.9$$

10. Money deposited in a certain bank doubles every 13 years. The bank compounds interest continuously. What annual interest rate does the bank offer?

Continuous compounding. In 13 years, money doubles.

$\cancel{P} \cdot e^{rt} = 2\cancel{P}$ if $t = 13$

$$e^{13r} = 2$$

$$\ln\left(e^{13r}\right) = \ln 2$$

$$13r = \ln 2$$

$$r = \frac{\ln 2}{13} = 0.05332 \text{ or } 5.332\%$$

11. How long will it take for a quantity of money A_0 to triple in value if it is invested at an annual interest rate r compounded continuously?

Continuous compounding. Want $A_0 \cdot e^{rt} = 3A_0$

$$e^{rt} = 3.$$

Find t in terms of r. $\ln(e^{rt}) = \ln 3$

$$rt = \ln 3$$

$$t = \frac{\ln 3}{r}$$

12. If an account that earns interest compounded continuously takes 12 years to double in value, how long will it take to triple in value?

12 yrs to double in value $\rightarrow$ $\cancel{P} \cdot e^{rt} = 2\cancel{P}$ if $t = 12$

$$e^{12r} = 2$$

$$\Rightarrow \ln(e^{12r}) = \ln 2$$

$$12r = \ln 2$$

$$r = \frac{\ln 2}{12}$$

What value of t will cause $\cancel{P} \cdot e^{rt} = 3\cancel{P}$?

$e^{rt} = 3 \Rightarrow \ln(e^{rt}) = \ln 3$

$$rt = \ln 3$$

$$t = \ln 3 \div r = \ln 3 \div \frac{\ln 2}{12}$$

$$= \ln 3 \cdot \frac{12}{\ln 2} = \frac{12 \ln 3}{\ln 2} = 19.02$$

13. The Morenos invest \$10,000 in an account that grows to \$12,000 in 5 years. What is the annual interest rate r if interest is compounded
a. Quarterly
b. Continuously

Principal $P = 10,000$. Future amount $B = 12,000$.
$t = 5$.

a. Compounded quarterly $\Rightarrow k = 4$. $B = P \cdot \left(1 + \frac{r}{k}\right)^{kt}$

$$12000 = 10,000\left(1 + \frac{r}{4}\right)^{20}$$

$$\frac{12,000}{10,000} = \left(1 + \frac{r}{4}\right)^{20} \Rightarrow \frac{12}{10} = \frac{6}{5} = \left(1 + \frac{r}{4}\right)^{20}$$

$$\Rightarrow \left(\frac{6}{5}\right)^{1/20} = 1 + \frac{r}{4}$$

$$\left(\frac{6}{5}\right)^{1/20} - 1 = \frac{r}{4} \Rightarrow r = 4\left(\left(\frac{6}{5}\right)^{1/20} - 1\right)$$

$$= 0.03663 \text{ or } 3.663\%$$

b. $B = Pe^{rt} \Rightarrow 12,000 = 10,000e^{5r}$

$$\Rightarrow \frac{12,000}{10,000} = e^{5r} \Rightarrow \frac{6}{5} = e^{5r} \Rightarrow \ln\left(\frac{6}{5}\right) = \ln\left(e^{5r}\right)$$

$$\Rightarrow \ln\left(\frac{6}{5}\right) = 5r \Rightarrow r = \frac{1}{5}\ln\left(\frac{6}{5}\right) \text{ or } \frac{1}{5}(\ln 6 - \ln 5)$$

$$= 0.03646 \text{ or } 3.646\%$$

14. An economist has compiled this data on the gross domestic product (GDP) of a certain country:

Year	1995	2005
GDP (in billions)	100	180

Use this data to predict the GDP in the year 2015 if the GDP is growing:
a. Linearly, so that GDP $= at + b$.
b. Exponentially, so that GDP $= Ae^{kt}$.

Let t = # of years since 1995.
Let G = GDP in billions. Then: $t=0 \Rightarrow G=100$
$t=10 \Rightarrow G=180$

a. $G = at + b \Rightarrow 100 = a\cdot 0 + b \Rightarrow 100 = b$
$180 = a\cdot 10 + b \Rightarrow 180 = 10a + b$

So $b=100$ and then $180 = 10a + 100$ so $a = 8$.

Therefore $G = 8t + 100$. In 2015, we have $t=20$
so $G = 8\cdot 20 + 100 = 160 + 100 = 260$ (billion)

b. $G = Ae^{kt} \Rightarrow 100 = A\cdot e^{k\cdot 0} \Rightarrow 100 = A\cdot 1 \Rightarrow A = 100$
$180 = A\cdot e^{k\cdot 10}$

$\Rightarrow 180 = 100e^{10k} \Rightarrow \frac{180}{100} = e^{10k} \Rightarrow e^{10k} = \frac{18}{10} = \frac{9}{5}$

$\Rightarrow 10k = \ln\left(\frac{9}{5}\right) \Rightarrow k = \ln(9/5)/10$

Therefore $G = 100\cdot e^{(\ln(9/5)/10)t} = 100\cdot e^{\ln(9/5)\cdot t/10}$

$= 100\cdot\left(9/5\right)^{t/10}$. So, if $t=20$, then $G = 100\cdot\left(9/5\right)^{20/10}$

$= 100\cdot\left(9/5\right)^2 = 100\cdot\frac{81}{25} = 324$ (billion)

15. Tests of an artifact discovered at the Debert site in Nova Scotia show that 28% of the original ^{14}C is still present. Approximately how old is the artifact? (The half-life of ^{14}C is 5,730 years.)

Exponential decay: $Q = Q_0 \cdot e^{-kt}$

Half-life of 5730 years $\Rightarrow \frac{1}{2} Q_0 = Q_0 \cdot e^{-k \cdot 5730}$

$$\Rightarrow \frac{1}{2} = e^{-5730k} \Rightarrow 2 = e^{5730k}$$

$$\ln 2 = \ln(e^{5730k})$$

$$\ln 2 = 5730k$$

$$\frac{\ln 2}{5730} = k$$

Current amount is 28% of starting amount. $Q = 0.28 Q_0$

$$0.28 Q_0 = Q_0 \cdot e^{-kt}$$

$$0.28 = e^{-kt}$$

$$\ln(0.28) = \ln(e^{-kt})$$

$$\ln(0.28) = -kt$$

$$-\frac{\ln(0.28)}{k} = t$$

$$t = -\ln(0.28) \div k$$

$$= -\ln(0.28) \div \frac{\ln 2}{5730}$$

$$= -\ln(0.28) \cdot \frac{5730}{\ln 2}$$

$$\approx 10,523$$

The artifact is about 10,523 years old.

16. A medical student studying the growth of bacteria in a certain culture has compiled this data:

Number of minutes	0	20
Number of bacteria	6,000	9,000

Use this data to find an exponential function of the form $Q(t) = Q_0 e^{kt}$ expressing the number of bacteria in the culture as a function of time. How many bacteria are present after 1 hour?

$Q_0 = 6000$. If $t = 20$, then $Q = 9000$.

$$9000 = 6000 \cdot e^{k \cdot 20} \Rightarrow \frac{9000}{6000} = e^{20k} \qquad \frac{9000}{6000} = \frac{9}{6} = \frac{3}{2}$$

$$\frac{3}{2} = e^{20k}$$

$$\ln(3/2) = \ln(e^{20k}) \Rightarrow \ln(3/2) = 20k$$

$$\frac{\ln(3/2)}{20} = k \Rightarrow k \approx 0.02027$$

Our function modeling the # of bacteria is $Q(t) = 6000 e^{0.02027t}$.

Note 1 hour = 60 minutes and we have been measuring time in minutes.

# of bacteria after 1 hour : $Q(60) = 6000 e^{0.02027 \cdot 60}$

$= 20,246$

$$Q = Q_0 \cdot e^{-kt}$$

17. The half-life of radium is 1,690 years. How long will it take for a 50-gram sample of radium to be reduced to 5 grams?

Half-life of 1690 years $\Rightarrow \frac{1}{2} Q_0 = Q_0 \cdot e^{-k \cdot 1690}$

$\Rightarrow \frac{1}{2} = e^{-1690k} \quad \Rightarrow 2 = e^{1690k}$

$$\ln 2 = \ln\left(e^{1690k}\right)$$

$$\ln 2 = 1690k$$

$$\frac{\ln 2}{1690} = k$$

If initial amount is 50 g and current amount is 5 g

then $5 = 50 \cdot e^{-kt}$

$$\frac{5}{50} = e^{-kt}$$

$$\frac{1}{10} = e^{-kt}$$

$$10 = e^{kt}$$

$$\ln(10) = \ln\left(e^{kt}\right)$$

$$\ln 10 = kt$$

$$\frac{\ln 10}{k} = t$$

$$t = \ln(10) \div k$$

$$= \ln(10) \div \frac{\ln 2}{1690}$$

$$= \ln(10) \cdot \frac{1690}{\ln 2}$$

$$\approx 5614$$

It will take about 5614 years.

18. Differentiate the function.

$$f(x) = e^{x^2+2x-1}$$ Use chain rule

$$f = e^u \text{ and } u = x^2 + 2x - 1$$

$$\frac{df}{du} = e^u \text{ and } \frac{du}{dx} = 2x + 2$$

$$f'(x) = \frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = e^u \cdot (2x+2) = e^{x^2+2x-1} \cdot (2x+2)$$

19. Differentiate the function.

$$f(x) = (x^2 + 3x + 5)e^{6x}$$ Use product rule

$$f'(x) = (x^2+3x+5)' e^{6x} + (x^2+3x+5)(e^{6x})'$$

$$= (2x+3)e^{6x} + (x^2+3x+5)e^{6x} \cdot 6$$ by chain rule

which can also be written

$$(2x+3)e^{6x} + (6x^2+18x+30)e^{6x}$$

or $(6x^2+20x+33)e^{6x}$

20. Differentiate the function.

$$f(x) = xe^{-x^2}$$

$$f'(x) = (x)' e^{-x^2} + x\left(e^{-x^2}\right)'$$

$$= 1 e^{-x^2} + x \cdot e^{-x^2} \cdot (-2x) \quad \text{using chain rule}$$

$$= 1 e^{-x^2} - 2x^2 e^{-x^2}$$

$$\text{or } (1 - 2x^2) e^{-x^2}$$

21. Differentiate the function.

$$f(x) = (1 - 3e^x)^2$$

$$f'(x) = 2(1 - 3e^x) \cdot (1 - 3e^x)' \quad \text{using chain rule}$$

$$= 2(1 - 3e^x) \cdot (-3e^x)$$

$$\text{or } -6e^x(1 - 3e^x)$$

22. Differentiate the function.

$$f(x) = \sqrt{1+e^x} = (1+e^x)^{1/2}$$

$$f'(x) = \frac{1}{2}(1+e^x)^{-1/2} \cdot (1+e^x)'$$ using chain rule

$$= \frac{1}{2}(1+e^x)^{-1/2} \cdot e^x$$

which can be written various ways, e.g. $\dfrac{e^x}{2\sqrt{1+e^x}}$

23. Differentiate the function.

$$f(x) = e^{\sqrt{3x}}$$

$f = e^u$ and $u = \sqrt{3x} = \sqrt{3}\cdot\sqrt{x} = \sqrt{3}\cdot x^{1/2}$

$\dfrac{df}{du} = e^u$ $\qquad \dfrac{du}{dx} = \sqrt{3}\cdot\dfrac{1}{2}x^{-1/2} = \dfrac{\sqrt{3}}{2}x^{-1/2}$ or $\dfrac{\sqrt{3}}{2\sqrt{x}}$

$$f'(x) = \frac{df}{dx} = \frac{df}{du}\cdot\frac{du}{dx} = e^u \cdot \frac{\sqrt{3}}{2\sqrt{x}}$$

$$= e^{\sqrt{3x}} \cdot \frac{\sqrt{3}}{2\sqrt{x}}$$

or $\dfrac{\sqrt{3}\cdot e^{\sqrt{3x}}}{2\sqrt{x}}$

24. Differentiate the function.

$$f(x) = e^{1/x}$$

$$f = e^u \quad \text{and} \quad u = \frac{1}{x} = x^{-1}$$

$$\frac{df}{du} = e^u \qquad \frac{du}{dx} = -1x^{-2}$$

$$f'(x) = \frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = e^u \cdot (-1)x^{-2}$$

$$= e^{1/x} \cdot (-1) \cdot \frac{1}{x^2} \quad \text{or} \quad \frac{-e^{1/x}}{x^2}$$

25. Differentiate the function.

$$f(x) = e^x \ln x$$

$$f'(x) = (e^x)' \ln x + e^x (\ln x)'$$

$$= e^x \ln x + e^x \cdot \frac{1}{x}$$

$$\text{or} \quad e^x \cdot \left(\ln x + \frac{1}{x}\right)$$

$$\text{or} \quad e^x \cdot \frac{x \ln x + 1}{x}$$

26. Differentiate the function.

$$F(x) = \ln(2x^3 - 5x + 1)$$

$F = \ln u$ and $u = 2x^3 - 5x + 1$

$\dfrac{dF}{du} = \dfrac{1}{u}$ $\qquad \dfrac{du}{dx} = 6x^2 - 5$

$$F'(x) = \frac{dF}{dx} = \frac{dF}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot (6x^2 - 5)$$

$$= \frac{1}{2x^3 - 5x + 1} \cdot (6x^2 - 5) = \frac{6x^2 - 5}{2x^3 - 5x + 1}$$

27. Differentiate the function.

$$f(x) = \ln(e^{-x} + x)$$

$f = \ln u$ and $u = e^{-x} + x$

$\dfrac{df}{du} = \dfrac{1}{u}$ $\qquad \dfrac{du}{dx} = -e^{-x} + 1$

$$f'(x) = \frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot (-e^{-x} + 1)$$

$$= \frac{1}{e^{-x} + x} \cdot (-e^{-x} + 1) = \frac{-e^{-x} + 1}{e^{-x} + x} \text{ or } \frac{1 - e^{-x}}{e^{-x} + x}$$

28. Differentiate the function.

$$g(u) = \ln(u + \sqrt{u^2+1})$$

$$g = \ln w \quad \text{and} \quad w = u + \sqrt{u^2+1}$$

$$w = u + (u^2+1)^{1/2}$$

$$\frac{dg}{dw} = \frac{1}{w} \qquad \frac{dw}{du} = 1 + \frac{1}{2}(u^2+1)^{-1/2} \cdot 2u$$

$$= 1 + \frac{u}{\sqrt{u^2+1}}$$

$$g'(u) = \frac{dg}{du} = \frac{dg}{dw} \cdot \frac{dw}{du} = \frac{1}{w} \cdot \left(1 + \frac{u}{\sqrt{u^2+1}}\right)$$

$$= \frac{1}{u+\sqrt{u^2+1}} \left(1 + \frac{u}{\sqrt{u^2+1}}\right)$$

29. Differentiate the function.

$$L(x) = \ln\left(\frac{x^2+2x-3}{x^2+2x+1}\right)$$

$$L(x) = \ln(x^2+2x-3) - \ln(x^2+2x+1)$$

$$L'(x) = \frac{1}{x^2+2x-3} \cdot (2x+2) - \frac{1}{x^2+2x+1} \cdot (2x+2)$$

30. Find the **second** derivative of the function.

$$f(x) = e^{2x} + 2e^{-x}$$

$$f'(x) = e^{2x}\cdot(2x)' + 2e^{-x}\cdot(-x)' \quad \text{by chain rule}$$

$$= e^{2x}\cdot 2 + 2e^{-x}\cdot(-1)$$

$$= 2e^{2x} - 2e^{-x}$$

$$f''(x) = 2e^{2x}\cdot(2x)' - 2e^{-x}\cdot(-x)' \quad \text{by chain rule}$$

$$= 2e^{2x}\cdot 2 - 2e^{-x}\cdot(-1)$$

$$= 4e^{2x} + 2e^{-x}$$

31. Find the **second** derivative of the function.

$$f(x) = \ln(2x) + x^2$$

Could use rules of logarithms first

$$f(x) = \ln 2 + \ln x + x^2$$

$$f'(x) = 0 + \frac{1}{x} + 2x$$

$$= x^{-1} + 2x$$

$$f''(x) = -1x^{-2} + 2$$

or $-\dfrac{1}{x^2} + 2$

or $-\dfrac{1}{x^2} + \dfrac{2x^2}{x^2} = \dfrac{2x^2-1}{x^2}$