32. Use logarithmic differentiation to find the derivative $f'(x)$.

$$f(x) = \frac{(x+2)^5}{(3x-5)^{1/6}}$$

$$\ln f(x) = \ln\left(\frac{(x+2)^5}{(3x-5)^{1/6}}\right)$$

$$\ln f(x) = \ln\left((x+2)^5\right) - \ln\left((3x-5)^{1/6}\right)$$

$$\ln f(x) = 5\ln(x+2) - \frac{1}{6}\ln(3x-5)$$

Now take $\frac{d}{dx}$ of both sides, using chain rule as appropriate

$$\frac{1}{f(x)} \cdot f'(x) = 5 \cdot \frac{1}{x+2} \cdot (x+2)' - \frac{1}{6} \cdot \frac{1}{3x-5} \cdot (3x-5)'$$

$$\frac{1}{f(x)} \cdot f'(x) = 5 \cdot \frac{1}{x+2} \cdot 1 - \frac{1}{6} \cdot \frac{1}{3x-5} \cdot 3$$

$$\frac{1}{f(x)} \cdot f'(x) = \frac{5}{x+2} - \frac{1}{2} \cdot \frac{1}{3x-5}$$

$$f'(x) = f(x) \cdot \left(\frac{5}{x+2} - \frac{1}{2} \cdot \frac{1}{3x-5}\right)$$

or $$f'(x) = \frac{(x+2)^5}{(3x-5)^{1/6}}\left(\frac{5}{x+2} - \frac{1}{2} \cdot \frac{1}{3x-5}\right)$$

33. Use logarithmic differentiation to find the derivative $f'(x)$.

$$f(x) = \left(\frac{2x+1}{1-3x}\right)^{1/4}$$

$$\ln f(x) = \ln\left(\left(\frac{2x+1}{1-3x}\right)^{1/4}\right)$$

$$\ln f(x) = \frac{1}{4}\ln\left(\frac{2x+1}{1-3x}\right)$$

$$\ln f(x) = \frac{1}{4}\left(\ln(2x+1) - \ln(1-3x)\right)$$

$$\ln f(x) = \frac{1}{4}\ln(2x+1) - \frac{1}{4}\ln(1-3x)$$

Now take $\frac{d}{dx}$ of both sides

$$\frac{1}{f(x)} \cdot f'(x) = \frac{1}{4}\cdot\frac{1}{2x+1}\cdot(2x+1)' - \frac{1}{4}\cdot\frac{1}{1-3x}\cdot(1-3x)'$$

$$\frac{1}{f(x)} \cdot f'(x) = \frac{1}{4}\cdot\frac{1}{2x+1}\cdot 2 - \frac{1}{4}\cdot\frac{1}{1-3x}\cdot(-3)$$

$$\frac{1}{f(x)} \cdot f'(x) = \frac{1}{2}\cdot\frac{1}{2x+1} + \frac{3}{4}\cdot\frac{1}{1-3x}$$

$$f'(x) = f(x)\cdot\left(\frac{1}{2}\cdot\frac{1}{2x+1} + \frac{3}{4}\cdot\frac{1}{1-3x}\right)$$

34. Use logarithmic differentiation to find the derivative $f'(x)$.

$$f(x) = (x+1)^3(6-x)^2(2x+1)^{1/3}$$

$$\ln f(x) = \ln\left((x+1)^3(6-x)^2(2x+1)^{1/3} \right)$$

$$\ln f(x) = \ln\left((x+1)^3\right) + \ln\left((6-x)^2\right) + \ln\left((2x+1)^{1/3}\right)$$

$$\ln f(x) = 3\ln(x+1) + 2\ln(6-x) + \frac{1}{3}\ln(2x+1)$$

$$\frac{1}{f(x)} \cdot f'(x) = 3 \cdot \frac{1}{x+1}(x+1)' + 2 \cdot \frac{1}{6-x}(6-x)' + \frac{1}{3} \cdot \frac{1}{2x+1} \cdot (2x+1)'$$

$$\frac{1}{f(x)} \cdot f'(x) = 3 \cdot \frac{1}{x+1} \cdot 1 + 2 \cdot \frac{1}{6-x} \cdot (-1) + \frac{1}{3} \cdot \frac{1}{2x+1} \cdot 2$$

$$\frac{1}{f(x)} \cdot f'(x) = 3 \cdot \frac{1}{x+1} - 2 \cdot \frac{1}{6-x} + \frac{2}{3} \cdot \frac{1}{2x+1}$$

$$f'(x) = f(x) \cdot \left(3 \cdot \frac{1}{x+1} - 2 \cdot \frac{1}{6-x} + \frac{2}{3} \cdot \frac{1}{2x+1} \right)$$

35. Use logarithmic differentiation to find the derivative $f'(x)$.

$$f(x) = \frac{e^{-3x}\sqrt{2x-5}}{(6-5x)^4}$$

$$\ln f(x) = \ln\left(\frac{e^{-3x}\sqrt{2x-5}}{(6-5x)^4}\right)$$

$$\ln f(x) = \ln\left(e^{-3x}\right) + \ln\left(\sqrt{2x-5}\right) - \ln\left((6-5x)^4\right)$$

$$\ln f(x) = -3x + \frac{1}{2}\ln(2x-5) - 4\ln(6-5x)$$

$$\frac{1}{f(x)} \cdot f'(x) = -3 + \frac{1}{2} \cdot \frac{1}{2x-5} \cdot 2 - 4 \cdot \frac{1}{6-5x} \cdot (-5)$$

$$\frac{1}{f(x)} \cdot f'(x) = -3 + \frac{1}{2x-5} + \frac{20}{6-5x}$$

$$f'(x) = f(x) \cdot \left(-3 + \frac{1}{2x-5} + \frac{20}{6-5x}\right)$$

36. Use logarithmic differentiation to find the derivative $f'(x)$.

$$f(x) = 5^{x^2}$$

$$\ln f(x) = \ln\left(5^{x^2}\right)$$

$$\ln f(x) = x^2 \underbrace{\ln 5}_{\text{constant}}$$

Now take $\frac{d}{dx}$ of both sides

$$\frac{1}{f(x)} \cdot f'(x) = 2x \cdot \ln 5$$

$$f'(x) = f(x) \cdot 2x \cdot \ln 5$$

or $f'(x) = 5^{x^2} \cdot 2x \cdot \ln 5$

or $2 \ln 5 \cdot x \cdot 5^{x^2}$

37. A certain industrial machine depreciates so that its value after t years becomes $Q(t) = 20,000e^{-0.4t}$ dollars.
a. At what rate is the value of the machine changing with respect to time after 5 years?
b. At what percentage rate is the value of the machine changing with respect to time after t years? Does this percentage rate depend on t or is it constant?

Given: Value $= Q(t) = 20,000\, e^{-0.4t}$ where t is time.

(a) asks for rate of change of value with respect to time

which is $\dfrac{dQ}{dt} = Q'(t) = 20,000 \cdot e^{-0.4t} \cdot (-0.4)$

$= -8,000 \cdot e^{-0.4t}$.

After 5 years, the desired rate of change is

$Q'(5) = -8000 \cdot e^{-0.4 \times 5} = -8000 \cdot e^{-2} = -1082.68$
dollars per year

(b) percentage rate of change after t years

is $100 \dfrac{Q'(t)}{Q(t)} = 100 \cdot \dfrac{-8000\, e^{-0.4t}}{20,000\, e^{-0.4t}} = 100 \cdot \dfrac{-8000}{20000}$

$= 100 \cdot \dfrac{-8}{20} = -40$ percent

this is constant (does not depend on t).

It's always depreciating at an instantaneous rate of 40% of its current value per year

38. The total number of hamburgers sold by a national fast-food chain is growing exponentially. If 4 billion had been sold by 2005 and 12 billion had been sold by 2010, how many will have been sold by 2015?

Total # of hamburgers is an exponential function of time.

Let Q = total # of hamburgers IN BILLIONS and let t = # of years since 2005

Then $Q(t) = Q_0 \cdot e^{kt}$ for some constants Q_0 and k

2005 $\to$ $t=0$ and total # of burgers is 4 (billion)

so $Q(0) = 4$ i.e. $Q_0 \cdot e^{k \cdot 0} = 4$

i.e. $Q_0 \cdot 1 = 4 \Rightarrow Q_0 = 4$

2010 $\to$ $t=5$ and total # of burgers is 12 (billion)

so $Q(5) = 12$ i.e. $Q_0 \cdot e^{k \cdot 5} = 12$

$4 \cdot e^{5k} = 12$

$\Rightarrow e^{5k} = 3 \Rightarrow 5k = \ln 3 \Rightarrow k = (\ln 3)/5$

Thus, $Q(t) = 4 \cdot e^{t(\ln 3)/5}$

sold by 2015 $= Q(10) = 4 \cdot e^{10(\ln 3)/5} = 4 \cdot e^{2 \ln 3}$

$= 4 \cdot (e^{\ln 3})^2 = 4 \cdot 3^2 = 4 \cdot 9 = 36$ (billion)

BTW Notice 4 billion $\xrightarrow{\times 3}$ 12 billion $\xrightarrow{\times 3}$ 36 billion

2005 $\xrightarrow[\text{5 years}]{}$ 2010 $\xrightarrow[\text{5 years}]{}$ 2015

39. Once the initial publicity surrounding the release of a new book is over, sales of the hardcover edition tend to decrease exponentially. At the time publicity was discontinued, a certain book was experiencing sales of 25,000 copies per month. One month later, sales of the book had dropped to 10,000 copies per month. What will the sales be after one more month?

Sales as a function of time is exponentially decreasing

Sales $= S(t) = S_0 \cdot e^{-kt}$ where t = time in months

Measure sales in thousands

Given: $S(0) = 25$ (thousand) and $S(1) = 10$ (thousand)

$$S_0 \cdot e^{-k \cdot 0} = 25 \qquad S_0 \cdot e^{-k \cdot 1} = 10$$
$$S_0 \cdot 1 = 25 \qquad 25 \cdot e^{-k} = 10$$
$$S_0 = 25 \qquad e^{-k} = \frac{10}{25} = \frac{2}{5}$$

$\Rightarrow -k = \ln(2/5)$

So $S(t) = S_0 \cdot e^{-kt} = 25 \cdot e^{t \ln(2/5)}$

We want $S(2)$, which is $25 \cdot e^{2\ln(2/5)}$

$= 25 \cdot \left(e^{\ln(2/5)}\right)^2 = 25 \cdot (2/5)^2 = 25 \cdot \frac{4}{25} = 4$

The sales after one more month has passed (i.e. two months have passed altogether) is 4,000.

40. It is estimated that the population of a certain country grows exponentially. If the population was 60 million in 1997 and 90 million in 2002, what will the population be in 2012?

Let $t = \#$ of years since 1997

Let P = population in millions

Exponential growth: $P(t) = P_0 \cdot e^{kt}$

$1997 \to t=0$, $2002 \to t=5$, $2012 \to t=15$

Given $P(0)=60$ and $P(5)=90$. Find $P(15)$

$P(0)=60 \Rightarrow P_0 \cdot e^{k \cdot 0} = 60 \Rightarrow P_0 \cdot 1 = 60 \Rightarrow P_0 = 60$

$$P(5) = 90 \Rightarrow P_0 \cdot e^{k \cdot 5} = 90$$
$$60 \cdot e^{5k} = 90$$
$$e^{5k} = \frac{90}{60} = \frac{9}{6} = \frac{3}{2} \Rightarrow 5k = \ln(3/2)$$

$\Rightarrow k = \ln(3/2)/5$

$$P(15) = P_0 \cdot e^{k \cdot 15} = 60 \cdot e^{15 \ln(3/2)/5} = 60\, e^{3 \ln(3/2)}$$

$$= 60 \cdot \left(e^{\ln(3/2)}\right)^3 = 60 \cdot (3/2)^3 = 60 \cdot 27/8 = 15 \cdot 27/2$$

$= 202.5$ (million)