MAC2233
Suggested problems on Chapter 5 material (integration)

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1. Evaluate the integral.

$$\int \left(\frac{1}{2y} - \frac{2}{y^2} + \frac{3}{\sqrt{y}}\right) dy$$

$$= \int \left(\frac{1}{2} \cdot \frac{1}{y} - 2 \cdot y^{-2} + 3 \cdot y^{-1/2} \right) dy$$

$$= \frac{1}{2} \ln y - 2 \cdot \frac{y^{-2+1}}{-2+1} + 3 \cdot \frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$= \frac{1}{2} \ln y - 2 \cdot \frac{y^{-1}}{-1} + 3 \cdot \frac{y^{1/2}}{1/2} + C$$

$$\text{or} \quad \frac{1}{2} \ln y + 2 \cdot \frac{1}{y} + 6 \cdot y^{1/2} + C$$

Note: $x\sqrt{x} = x^{1} \cdot x^{1/2} = x^{1+\frac{1}{2}} = x^{3/2}$

2. Evaluate the integral.

$$\int \left(\frac{e^x}{2} + x\sqrt{x}\right) dx = \int \left(\frac{1}{2} e^x + x^{3/2}\right) dx$$

$$= \frac{1}{2} e^x + \frac{x^{5/2}}{5/2} + C$$

because $\frac{3}{2} + 1 = \frac{5}{2}$

or $\frac{1}{2} e^x + \frac{2}{5} x^{5/2} + C$

3. Evaluate the integral.

$$\int \left(2e^u + \frac{6}{u} + \ln 2\right) du$$

$$\int \left(2 \cdot e^u + 6 \cdot \frac{1}{u} + \underbrace{\ln 2}_{\substack{\text{Constant} \\ \text{Just a number}}}\right) du$$

$$= 2 \cdot e^u + 6 \ln u + (\ln 2) \cdot u + C$$

4. Evaluate the integral.

$$\int \left(\frac{x^2+2x+1}{x^2}\right) dx$$

$$= \int \left(\frac{x^2}{x^2} + \frac{2x}{x^2} + \frac{1}{x^2} \right) dx$$

$$= \int \left(1 + 2 \cdot \frac{1}{x} + x^{-2} \right) dx$$

5. Evaluate the integral.

$$\int (x^3 - 2x^2)\left(\frac{1}{x} - 5\right) dx$$ Expand first

$$\int \left(x^3 \cdot \frac{1}{x} - x^3 \cdot 5 - 2x^2 \cdot \frac{1}{x} + 2x^2 \cdot 5 \right) dx$$

$$= \int \left(x^2 - 5x^3 - 2x + 10x^2 \right) dx$$

$$= \int \left(11x^2 - 5x^3 - 2x \right) dx$$

$$= 11 \cdot \frac{x^3}{3} - 5 \cdot \frac{x^4}{4} - 2 \cdot \frac{x^2}{2} + C$$

$$\int \left(2 \cdot \frac{1}{x} - x^{-2}\right) dx$$

$$= 2 \cdot \ln x - \frac{x^{-2+1}}{-2+1}$$

6. Solve the initial value problem.

$$\frac{dy}{dx} = \frac{2}{x} - \frac{1}{x^2} \qquad \text{where } y = -1 \text{ when } x = 1$$

$$\frac{dy}{dx} = 2 \cdot \frac{1}{x} - x^{-2}$$

take antiderivative

$$\Rightarrow \quad y = 2 \ln x - \frac{x^{-1}}{-1} + C$$

$$y = 2 \ln x + \frac{1}{x} + C$$

Given: $y = -1$ when $x = 1$. We can use this to find C

$$-1 = 2 \underbrace{\ln 1}_{0} + \frac{1}{1} + C$$

$$-1 = 1 + C \qquad \Rightarrow C = -2$$

Final answer: $y = 2 \ln x + \frac{1}{x} - 2$

7. Given $f'(x) = e^{-x} + x^2$ and $f(0) = 4$, find $f(x)$.

$\downarrow$ take antiderivative

$$f(x) = -e^{-x} + \frac{x^3}{3} + C$$

Given $f(0)=4$. This will help us find C.

$$4 = f(0) = -e^{-0} + \frac{0^3}{3} + C$$

$$4 = -1 + 0 + C$$

$$\Rightarrow C = 5$$

Final answer: $f(x) = -e^{-x} + \frac{x^3}{3} + 5$

8. A manufacturer estimates that the marginal cost of producing q units of a certain commodity is $C'(q) = 3q^2 - 24q + 48$ dollars per unit. If the cost of producing 10 units is \$5,000, what is the cost of producing 30 units?

Given: $C'(q) = 3q^2 - 24q + 48$

Take antiderivative: $C(q) = 3 \cdot \frac{q^3}{3} - 24 \frac{q^2}{2} + 48q + K$ $= q^3 - 12q^2 + 48q + K$

where K is a constant

We don't know K yet, but we can find it.

Given: When the quantity is 10, the cost is 5000.

That is, $C(10) = 5000$. (when $q = 10$, C is 5000)

So $5000 = C(10) = \cancel{3} \cdot \frac{10^3}{\cancel{3}} - 24 \cdot \frac{10^2}{2} + 48 \cdot 10 + K$

$5000 = 1000 - \underbrace{12 \cdot 100}_{1200} + 480 + K$

$5000 = 280 + K$ so $K = 4720$

So $C(q) = q^3 - 12q^2 + 48q + 4720$

So cost of producing 30 units is $C(30)$, which is

$C(30) = 30^3 - 12 \cdot 30^2 + 48 \cdot 30 + 4720$

$= 27{,}000 - 12 \cdot 900 + 1440 + 4720 = \ldots$ arithmetic $= 22{,}360$

9. A manufacturer estimates marginal revenue to be $R'(q) = 100q^{-1/2}$ dollars per unit when the level of production is q units. The corresponding marginal cost has been found to be $0.4q$ dollars per unit. Suppose the manufacturer's profit is \$520 when the level of production is 16 units. What is the manufacturer's profit when the level of production is 25 units?

Given: $R'(q) = 100q^{-1/2}$ (marginal revenue)

Given: $C'(q) = 0.4q$ (marginal cost)

We know $P(q) = R(q) - C(q)$ (profit = revenue − cost)

and $P'(q) = R'(q) - C'(q)$

So $P'(q) = 100q^{-1/2} - 0.4q$

Take antiderivative: $P(q) = 100\dfrac{q^{1/2}}{1/2} - 0.4\dfrac{q^2}{2} + K$

$= 200q^{1/2} - 0.2q^2 + K$

Given: Profit is 520 when production is 16. $P(16) = 520$

$520 = P(16) = 200 \cdot 16^{1/2} - 0.2 \cdot 16^2 + K = 200 \cdot 4 - 0.2 \cdot 256 + K$

$= 800 - 51.2 + K$ so $K = 520 - 800 + 51.2 = -228.8$

So $P(q) = 200q^{1/2} - 0.2q^2 - 228.8$

Then $P(25) = 200 \cdot 25^{1/2} - 0.2 \cdot 25^2 - 228.8$

$= 200 \cdot 5 - 0.2 \times 625 - 228.8 = 1000 - 125 - 228.8 = 646.2$

10. The marginal profit of a certain commodity is $P'(q) = 100 - 2q$ when q units are produced. When 10 units are produced, the profit is \$700.
a. Find the profit function $P(q)$.
b. What production level q results in maximum profit? What is the maximum profit?

(a) $P'(q) = 100 - 2q$

$\Rightarrow P(q) = 100q - q^2 + K$

Given $P(10) = 700$, so $700 = 100 \cdot 10 - 10^2 + K$
$= 1000 - 100 + K = 900 + K$

So $K = -200$, so profit function is
$$P(q) = 100q - q^2 - 200$$

(b) To maximize profit, set $P' = 0$

$100 - 2q = 0 \Rightarrow q = 50$

Notice also that if $q < 50$ then $P' = 100 - 2q$ is positive
(e.g. $P'(40) = 100 - 80$)
if $q > 50$ then $P' = 100 - 2q$ is negative
(e.g. $P'(60) = 100 - 120$)

So profit really is <u>maximized</u> when $q = 50$.

The maximum profit is $P(50) = 100 \cdot 50 - 50^2 - 200$
$= 5000 - 2500 - 200 = 2300$.

11. At a certain factory, when K thousand dollars is invested in the plant, the production Q is changing at a rate given by

$$Q'(K) = 200K^{-2/3}$$

units per thousand dollars invested. When \$8,000 is invested, the level of production is 5,500 units.
a. Find a formula for the level of production Q to be expected when K thousand dollars is invested.
b. How many units will be produced when \$27,000 is invested?
c. What capital investment K is required to produce 7,000 units?

(a) $Q'(K) = 200 \cdot K^{-2/3} \Rightarrow Q(K) = 200 \cdot \dfrac{K^{1/3}}{1/3} + C$

$= 600K^{1/3} + C.$

Given: When $K=8$, we have $Q = 5500$.

So $5500 = Q(8) = 600 \cdot 8^{1/3} + C = 600 \cdot 2 + C = 1200 + C$

so $C = 4300$. So $\boxed{Q(K) = 600K^{1/3} + 4300.}$

(b) \$27,000 invested $\Rightarrow K = 27$

$Q(27) = 600 \cdot 27^{1/3} + 4300 = 600 \cdot 3 + 4300$

$= 1800 + 4300 = \boxed{6100}$

(c) $Q(K) = 7000$. Find K. $600K^{1/3} + 4300 = 7000$

$\Rightarrow 600K^{1/3} = 2700$

$K^{1/3} = \dfrac{2700}{600} = \dfrac{27}{6} = \dfrac{9}{2}$

$K = \left(\dfrac{9}{2}\right)^3 = \dfrac{729}{8} = 91.125$ or $\boxed{\$91,125}$

12. Suppose the consumption function for a particular country is $c(x)$, where x is national disposable income. Then the **marginal propensity to consume** is $c'(x)$. Suppose x and c are both measured in billions of dollars and

$$c'(x) = 0.9 + 0.3\sqrt{x}.$$

If consumption is 10 billion dollars when $x = 0$, find $c(x)$.

$$c'(x) = 0.9 + 0.3x^{1/2}$$

$$\Rightarrow c(x) = 0.9x + 0.3\frac{x^{3/2}}{3/2} + K$$ where K is a constant

$$= 0.9x + 0.2x^{3/2} + K$$

Given: When $x=0$, c will be 10 (note c is measured in billions)

$$10 = c(0) = 0.9(0) + 0.2(0)^{3/2} + K = K$$

$$10 = K$$

$$c(x) = 0.9x + 0.2x^{3/2} + 10$$

13. A manufacturer estimates marginal revenue to be $200q^{-1/2}$ dollars per unit when the level of production is q units. The corresponding marginal cost has been found to be $0.4q$ dollars per unit. If the manufacturer's profit is \$2,000 when the level of production is 25 units, what is the profit when the level of production is 36 units?

Marginal revenue $= 200q^{-1/2}$ i.e. $R'(q) = 200q^{-1/2}$

Marginal cost $= 0.4q$ i.e. $C'(q) = 0.4q$

We know $P(q) = R(q) - C(q)$ and $P'(q) = R'(q) - C'(q)$

$$P'(q) = 200q^{-1/2} - 0.4q$$

$$\Rightarrow P(q) = 200\frac{q^{1/2}}{1/2} - 0.4\frac{q^2}{2} + K$$ where K is a constant

$$P(q) = 400q^{1/2} - 0.2q^2 + K$$

Given $P(25) = 2000 = 400 \cdot 25^{1/2} - 0.2 \times 25^2 + K$

$= 400 \cdot 5 - 0.2 \times 625 + K = 2000 - 125 + K \Rightarrow K = 125$

so $P(q) = 400q^{1/2} - 0.2q^2 + 125$

$$P(36) = 400 \cdot 36^{1/2} - 0.2 \cdot 36^2 + 125$$

$= 400 \cdot 6 - 0.2 \cdot 1296 + 125 = \ldots = 2265.8$

14. It is estimated that t months from now the population of a certain town will be increasing at the rate of $4+5t^{2/3}$ people per month. If the current population is 10,000, what will the population be 8 months from now?

Let $P(t)$ = population as a function of time

Given: <u>rate</u> of increase of population is $4+5t^{2/3}$

$$P'(t) = 4 + 5t^{2/3}$$

$$\Rightarrow P(t) = 4t + \frac{5t^{5/3}}{5/3} + C \qquad (C = \text{constant})$$

$$= 4t + \frac{3}{5} \cdot 5t^{5/3} + C$$

$$= 4t + 3t^{5/3} + C$$

$$10{,}000 = P(0) = 4 \cdot 0 + 3 \cdot 0^{5/3} + C$$

$$\Rightarrow C = 10{,}000$$

$$\Rightarrow P(t) = 4t + 3t^{5/3} + 10{,}000$$

so $P(8) = 4 \cdot 8 + 3 \cdot 8^{5/3} + 10{,}000$

$= 32 + 3 \cdot (8^{1/3})^5 + 10{,}000 = 32 + 3 \cdot 2^5 + 10{,}000$

$= \ldots = 10{,}128$