WRITE YOUR NAME:

MAC 2233 Section U02 Final Exam
Monday April 23rd
Total possible score: 30 points (3 points per page)

Question 1a. Find the limit.

$$\lim_{x\to 3} \frac{x-3}{x^2-9}$$

$$\lim_{x\to 3} \frac{x-3}{(x+3)(x-3)} = \lim_{x\to 3} \frac{1}{x+3} = \frac{1}{3+3}$$

$$= \frac{1}{6}$$

Question 1b. Find the limit.

$$\lim_{x\to\infty} \frac{3x^2+2x+7}{5x^2-8x+11} = \frac{3}{5}$$

$$\lim_{x\to\infty} \frac{3x^2+2x+7}{5x^2-8x+11} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{3+\frac{2}{x}+\frac{7}{x^2}}{5-\frac{8}{x}+\frac{11}{x^2}} = \frac{3+0+0}{5-0+0} = \frac{3}{5}$$

Question 2. Differentiate the function and simplify your answer.

$$f(x) = \frac{1}{\sqrt{5x^2+1}}$$

$$f(x) = (5x^2+1)^{-1/2}$$

$$f'(x) = \underbrace{-\frac{1}{2}(5x^2+1)^{-3/2}}_{\substack{\text{Took deriv. of outside}\\ \text{Left inside alone}}} \cdot \underbrace{10x}_{\text{Deriv. of inside}}$$

(chain rule)

$$f'(x) = -5x(5x^2+1)^{-3/2}$$

or $$\frac{-5x}{(5x^2+1)^{3/2}}$$

Question 3. The total cost of producing x hundred units of a certain commodity is $C(x) = 2x^2 + 3x + 50$ thousand dollars.

a. Use marginal analysis to estimate the cost of producing the ~~3rd~~ 300th unit.

b. What is the actual cost of producing the ~~3rd~~ 300th unit?

Because the numbers are messy, I decided to be flexible with grading this one.

$$C(x) = 2x^2 + 3x + 50$$

$$\Rightarrow C'(x) = 4x + 3$$

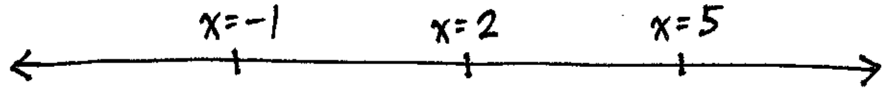
(a) Use marginal analysis to estimate cost of 3rd or 300th unit

$\rightarrow$ evaluate $C'(x)$ at appropriate value of x.

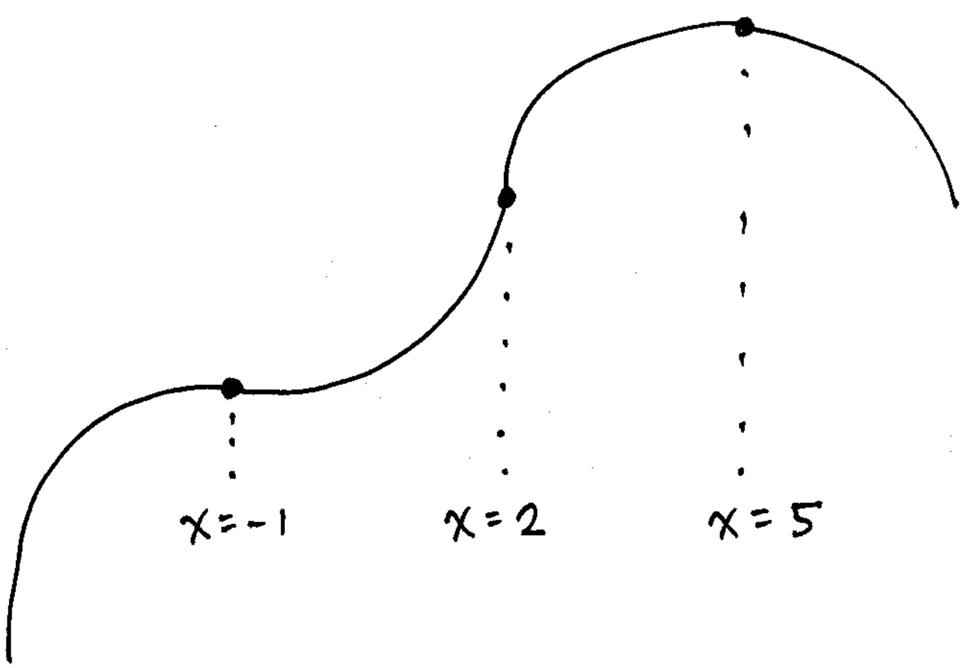
(b) Actual cost of producing 3rd or 300th unit

$\rightarrow$ evaluate the difference $C(x_2) - C(x_1)$ for appropriate values of x_1 and x_2.

Question 4. Sketch the graph of a function that has all the following properties:
$f'(x) > 0$ when $x < 5$
$f'(x) < 0$ when $x > 5$
$f''(x) < 0$ when $x < -1$ and when $x > 2$
$f''(x) > 0$ when $-1 < x < 2$



$f'>0$	$f'>0$	$f'>0$	$f'<0$
$f''<0$	$f''>0$	$f''<0$	$f''<0$
f increasing and conc down	f incr and conc up	f incr and conc down	f decreasing and concave down



Question 5. Determine where the function is increasing and decreasing, and where its graph is concave up and concave down. Find the relative extrema and inflection points, and sketch the graph of the function.

$$f(x) = x^4 - 4x^3 + 17$$

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$

$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

Increasing/decreasing?

0 3

\+ + + + 0 + + + + + + + x^2

\- - - - - - - - - 0 + + + $x-3$

$f'<0$ $f'<0$ $f'>0$

f decr f decr f incr

MINIMUM

Concave up or down?

0 2

\- - - - 0 + + + + + + + + x

\- - - - - - - - - 0 + + + + $x-2$

$f''>0$ $f''<0$ $f''>0$

f conc up f conc down f conc up

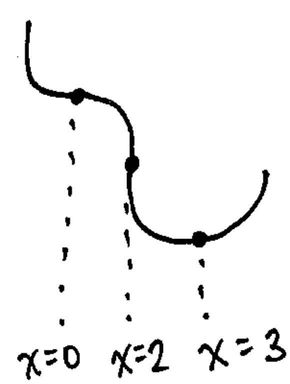
INFLECTION INFLECTION

So

0 2 3

decr decr decr incr

c.up c.down c.up c.up



Question 6. Find the absolute maximum and absolute minimum (if any) of the given function on the specified interval.

$$f(x) = \frac{x^2+8}{x-1}, \qquad 2 \le x \le 10$$

$$f'(x) = \frac{(x^2+8)'(x-1) - (x^2+8)(x-1)'}{(x-1)^2}$$

$$= \frac{2x(x-1) - (x^2+8)}{(x-1)^2} = \frac{2x^2 - 2x - x^2 - 8}{(x-1)^2} = \frac{x^2 - 2x - 8}{(x-1)^2}$$

$$= \frac{(x+2)(x-4)}{(x-1)^2}.$$

Critical numbers when top or bottom is 0.
$x = -2$, $x = 4$, $x = 1$

The only critical number <u>in the interval</u> is $x = 4$. Check $x=4$ and both endpoints.

$$f(2) = \frac{2^2+8}{2-1} = \frac{4+8}{1} = \frac{12}{1} = 12$$

$$f(4) = \frac{4^2+8}{4-1} = \frac{16+8}{3} = \frac{24}{3} = 8$$

$$f(10) = \frac{10^2+8}{10-1} = \frac{100+8}{9} = \frac{108}{9} = 12$$

So the absolute max is 12 and the absolute min is 8.

Question 7. A 500 room hotel in Miami Beach is filled to capacity every night at a \$150 rate per room. Management estimates that for each \$20 increase in the rate per room, 10 fewer rooms will be rented each day. The cost to service a rented room is \$8 per day. How much should management charge for each room to maximize profit? Be sure to use calculus to answer this question, identifying a variable x and expressing the quantity to be maximized as a function of x.

Given: If rate per room is 150, then # of rooms is 500.

Also given: if rate per room is $150+20x$, # of rooms is $500-10x$.

So: Total revenue $= \underbrace{(150+20x)}_{\text{revenue per room}}\underbrace{(500-10x)}_{\text{\# of rooms rented}}$

And total cost $= \underbrace{8}_{\text{cost per room}}\underbrace{(500-10x)}_{\text{\# of rooms rented}}$

Profit = revenue − cost $= (150+20x)(500-10x) - 8(500-10x)$

$$P(x) = \underbrace{(142+20x)}_{\text{profit per room}}\underbrace{(500-10x)}_{\text{\# of rooms rented}} = 2(71+10x)\cdot 10(50-x)$$

$$P(x) = 20(71+10x)(50-x) = 20(3550+500x-71x-10x^2)$$

$$= 20(3550+429x-10x^2) \Rightarrow P'(x) = 20(429-20x)$$

Critical numbers? $429-20x=0 \Rightarrow 20x = 429 \Rightarrow 2x = 42.9$

$x = 21.45$

If $x < 21.45$ then $P'(x) > 0$

If $x > 21.45$ then $P'(x) < 0$

So profit is MAXIMIZED when $x = 21.45$ i.e. $20x = 429$

We should charge $150 + 429 = 579$ dollars per room.

Question 8. Find the derivative of the function, and find **all** critical numbers (that is, all x values that make $f'(x)$ zero or undefined).

$$f(x) = (x-2)^3(x-9)^4$$

$$f'(x) = \left((x-2)^3\right)'(x-9)^4 + (x-2)^3\left((x-9)^4\right)'$$

$$= 3(x-2)^2(x-9)^4 + (x-2)^3 \cdot 4(x-9)^3$$

$$= (x-2)^2(x-9)^3\left[\underbrace{3(x-9)}_{3x-27} + \underbrace{4(x-2)}_{4x-8}\right]$$

$$= (x-2)^2(x-9)^3(7x-35)$$

Critical numbers when $x-2=0$ or $x-9=0$ or $7x-35=0$

i.e. $x=2$, $x=5$, $x=9$

Question 9. The quoted annual interest rate is 6% and it is compounded continuously.

a. How long will it take for an investment to double in value?
For full credit, tell me which of the following times is closest to the answer: 10 years, 15 years, 20 years, or 25 years.

b. How long will it take for an investment to triple in value?
For full credit, tell me which of the following times is closest to the answer: 10 years, 15 years, 20 years, or 25 years.

In this problem, you can use the fact that $\ln 2 \approx 0.69$ and $\ln 3 \approx 1.1$.

$r = 0.06$. Continuous compounding $\Rightarrow B = P \cdot e^{rt}$

$$B = P \cdot e^{0.06t}.$$

(a) Doubling in value $\Rightarrow B = 2P$

$$2P = P \cdot e^{0.06t} \Rightarrow 2 = e^{0.06t}$$

$$\ln 2 = \ln\left(e^{0.06t}\right)$$

$$\ln 2 = 0.06t$$

$$\frac{\ln 2}{0.06} = t \qquad \frac{0.69}{0.06} = \frac{69}{6} \approx 11.5$$

Closest to 10

(b) Tripling in value is similar: $3 = e^{0.06t}$

$$\ln 3 = 0.06t$$

$$\frac{\ln 3}{0.06} = t$$

$$\frac{1.10}{0.06} = \frac{110}{6} \approx 18.3$$ Closest to 20

Question 10. The resale value of an industrial machine decreases over a 10 year period at a rate that changes with time. When the machine is x years old, the rate at which its value is changing is $200(x-10)$ dollars per year. By how much does the machine depreciate during the second year?

Let value at time x be $V(x)$

Given $V'(x) = 200(x-10) = 200x - 2000$

Then $V(x) = 200\dfrac{x^2}{2} - 2000x + C$

$= 100x^2 - 2000x + C$

Value at end of second year is

$V(2) = 100 \cdot 2^2 - 2000 \cdot 2 + C$

$= 400 - 4000 + C = C - 3600$

Value at end of first year is

$V(1) = 100 \cdot 1^2 - 2000 \cdot 1 + C$

$= 100 - 2000 + C = C - 1900$

<u>Change</u> in value <u>during</u> second year is $V(2) - V(1)$

$= (C-3600) - (C-1900) = C - 3600 - C + 1900$

$= -1700$. The machine depreciates by \$1700.