MAC2241
Suggested problems on Chapter 4 material
(applications of derivatives)

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1. Find the intervals on which the function is increasing, decreasing, concave up, or concave down.

$$f(x) = 5 - 4x - x^2$$

$f'(x) = -4 - 2x$

$f''(x) = -2$ ← Always negative, so f is always concave down.

$f'(x) = 0$? $\quad -4 - 2x = 0$

$\quad -4 = 2x \Rightarrow x = -2$

$x = -2$

if x is here (e.g. $x = -3$) then f' is positive

if x is here (e.g. $x = 0$) then f' is negative

f is increasing on $(-\infty, -2)$

f is decreasing on $(-2, \infty)$

f is concave down on $(-\infty, \infty)$

(f is never concave up)

2. Find the intervals on which the function is increasing, decreasing, concave up, or concave down.

$$f(x) = (2x+1)^3$$

$$f'(x) = 3(2x+1)^2 \cdot 2 = 6(2x+1)^2$$

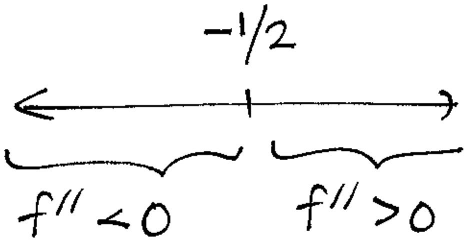
$$f''(x) = 6 \cdot 2(2x+1) \cdot 2 = 24(2x+1)$$

f' is never negative. (because of the square)

$f'' = 0$? When $2x + 1 = 0$

$$2x = -1$$

$$x = -1/2$$



f is always increasing

f is concave down on $(-\infty, -\frac{1}{2})$

f is concave up on $(-\frac{1}{2}, \infty)$

3. Find the intervals on which the function is increasing, decreasing, concave up, or concave down.

$$f(x) = 5 + 12x - x^3$$

$$f'(x) = 0 + 12\cdot 1 - 3x^2$$
$$= 12 - 3x^2$$
$$f''(x) = 0 - 3\cdot 2x = -6x$$

$$f'(x) = 12 - 3x^2 = 3(4 - x^2) = 3(2+x)(2-x)$$

$x=-2$ $\qquad$ $x=2$

$- - - - - 0 + + + + + + + + + +$ $\quad 2+x$

$+ + + + + + + + + + + + + + 0 - - - -$ $\quad 2-x$

$f' = $ neg $\cdot$ pos	$f' = $ pos $\cdot$ pos	$f' = $ pos $\cdot$ neg
f' neg	f' pos	f' neg
f decreasing	f increasing	f decreasing

$x=0$

$- - - - - - - 0 + + + + + + +$ $\quad x$

$- - - - - - - - - - - - - - -$ $\quad -6$

$f''(x) = -6x$

f'' is positive	f'' is negative
($-6x$ is positive)	($-6x$ is negative)
f is concave UP	f is concave DOWN

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

4. Find the intervals on which the function is increasing, decreasing, concave up, or concave down.

$$f(x) = \frac{x}{x^2+4}$$

$$f'(x) = \frac{(x)'(x^2+4) - (x)(x^2+4)'}{(x^2+4)^2}$$

$$= \frac{1(x^2+4) - x\cdot 2x}{(x^2+4)^2} = \frac{x^2+4-2x^2}{(x^2+4)^2} = \frac{4-x^2}{(x^2+4)^2}$$

$$f''(x) = \frac{(4-x^2)'(x^2+4)^2 - (4-x^2)\left((x^2+4)^2\right)'}{\left((x^2+4)^2\right)^2}$$

Must use chain rule! $(u^2)' = 2u\cdot u'$

$$= \frac{-2x(x^2+4)^2 - (4-x^2)\cdot 2(x^2+4)\cdot 2x}{(x^2+4)^4}$$

Then notice $2x$ and x^2+4 are common factors on top. Can also factor out a -1

$$= \frac{-2x(x^2+4)\left[(x^2+4) + \overbrace{(4-x^2)\cdot 2}^{8-2x^2}\right]}{(x^2+4)^4} = \frac{-2x(x^2+4)(12-x^2)}{(x^2+4)^4}$$

$$= \frac{-2x(12-x^2)}{(x^2+4)^3} \quad \text{or} \quad \frac{2x(x^2-12)}{(x^2+4)^3}$$

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$$f'(x) = \frac{4-x^2}{(x^2+4)^2}.$$

Where might f' change sign?

When $4-x^2=0$ or $(x^2+4)^2=0$

$x^2=4$

$x=-2$ or $x=2$

$x^2+4=0$

Never happens

$-2 \qquad 2$

$- - - - - - 0 + + + + 0 - - - - - \quad 4-x^2$

$+ + + + + + + + + + + + + + + + + + \quad (x^2+4)^2$

$f' = \frac{neg}{pos}$	$f' = \frac{pos}{pos}$	$f' = \frac{neg}{pos}$
f' neg	f' pos	f' neg
f decreasing	f increasing	f decreasing

$$f''(x) = \frac{2x(x^2-12)}{(x^2+4)^3}$$

Where might f'' change sign?

When $2x=0$ i.e. $x=0$

or when $x^2-12=0$ i.e. $x^2=12$

$x=\pm\sqrt{12}$

or when $(x^2+4)^3=0$ i.e. $x^2+4=0$

Never happens

$-\sqrt{12} \qquad 0 \qquad \sqrt{12}$

$- - - - - - - - - - 0 + + + + + + + + \quad 2x$

$+ + + + + 0 - - - - - - - - - - 0 + + + + \quad x^2-12$

$+ \quad (x^2+4)^3$

$f'' = \frac{neg \cdot pos}{pos}$	$f'' = \frac{neg \cdot neg}{pos}$	$f'' = \frac{pos \cdot neg}{pos}$	$f'' = \frac{pos \cdot pos}{pos}$
f'' neg	f'' pos	f'' neg	f'' pos
f conc down	f conc up	f conc down	f conc up

5. Find the intervals on which the function is increasing, decreasing, concave up, or concave down.

$$f(x) = x^{4/3} - x^{1/3}$$

$$f'(x) = \frac{4}{3}x^{1/3} - \frac{1}{3}x^{-2/3}$$

$$f''(x) = \frac{4}{3}\cdot\frac{1}{3}x^{-2/3} - \frac{1}{3}\cdot\frac{-2}{3}x^{-5/3}$$

$$f'(x) = \frac{4x^{1/3}}{3} - \frac{1}{3x^{2/3}} = \frac{4x}{3x^{2/3}} - \frac{1}{3x^{2/3}} = \frac{4x-1}{3x^{2/3}}$$

$$f''(x) = \frac{4}{9x^{2/3}} + \frac{2}{9x^{5/3}} = \frac{4x}{9x^{5/3}} + \frac{2}{9x^{5/3}} = \frac{4x+2}{9x^{5/3}}$$

$$f'(x) = \frac{4x-1}{3(x^{1/3})^2}$$

Top changes from neg to pos at $x = \frac{1}{4}$

Bottom is never negative because of the square

$$\underbrace{\longleftarrow\quad}_{\substack{f' \text{ neg} \\ f \text{ decreasing}}} \overset{1/4}{+} \underbrace{\quad\longrightarrow}_{\substack{f' \text{ pos} \\ f \text{ increasing}}}$$

$$f''(x) = \frac{2(2x+1)}{9(x^{1/3})^5}$$

Top changes from neg to pos at $x = -\frac{1}{2}$

Bottom changes from neg to pos at $x = 0$

	$-1/2$		0	
$2x+1$	$- - - -$ 0	$+ + + + +$	$+ +$	$+ + +$
$(x^{1/3})^5$	$- - - -$	$- - - -$	$-$ 0	$+ + +$
	f'' pos, f conc UP	f'' neg, f conc DOWN		f'' pos, f conc UP

6. Find all the critical points of the function.

$$f(x) = 4x^4 - 16x^2 + 17$$

$$f'(x) = 4 \cdot 4x^3 - 16 \cdot 2x$$
$$= 16x^3 - 32x$$
$$= 16x(x^2 - 2)$$

f' is never undefined

$f' = 0$ if $x = 0$ or $x^2 - 2 = 0$

$$x^2 = 2$$
$$x = \pm\sqrt{2}$$

The critical numbers are

$x = -\sqrt{2}$, $x = 0$, $x = \sqrt{2}$

7. Find all the critical points of the function.

$$f(x) = 3x^4 + 12x$$

$$f'(x) = 3 \cdot 4x^3 + 12 \cdot 1$$

$$= 12x^3 + 12$$

$$= 12(x^3 + 1)$$

f' is never undefined.

$f' = 0$? when $x^3 + 1 = 0$

$$x^3 = -1$$

$$\Rightarrow x = -1$$

The only critical number is $x = -1$.

8. Find all the critical points of the function.

$$f(x) = \frac{x+1}{x^2+3}$$

$$f'(x) = \frac{(x+1)'(x^2+3) - (x+1)(x^2+3)'}{(x^2+3)^2}$$

$$= \frac{1\cdot(x^2+3)-(x+1)\cdot 2x}{(x^2+3)^2} = \frac{x^2+3-2x^2-2x}{(x^2+3)^2} = \frac{-x^2-2x+3}{(x^2+3)^2}$$

$$= \frac{-(x^2+2x-3)}{(x^2+3)^2} = \frac{-(x+3)(x-1)}{(x^2+3)^2}$$

f' is undefined if denominator is zero. $(x^2+3)^2=0$

$x^2+3=0$

Never happens.

f' is zero if numerator is zero. $-(x+3)(x-1)=0$

$x+3=0$ or $x-1=0$

$x=-3$ or $x=1$

The critical numbers are $x=-3$ and $x=1$.

9. Find all the critical points of the function.

$$f(x) = (x^2 - 25)^{1/3}$$

To take the derivative, we must use the chain rule.

$$f = (x^2-25)^{1/3} \Rightarrow f = u^{1/3} \text{ and } u = x^2 - 25$$

$$\frac{df}{du} = \frac{1}{3}u^{-2/3} \qquad \frac{du}{dx} = 2x$$

$$f'(x) = \underbrace{\frac{1}{3}(x^2-25)^{-2/3}}_{\substack{\text{Took deriv. of outside} \\ \text{Left inside alone}}} \cdot \underbrace{2x}_{\text{Deriv. of inside}}$$

$$f'(x) = \frac{2x}{3(x^2-25)^{2/3}} \quad \text{or} \quad \frac{2x}{3\left((x^2-25)^{1/3}\right)^2}$$

Critical numbers are x values that make f' zero or undefined.

$f' = 0$ if $2x = 0$ i.e. if $x = 0$

f' is undefined if $x^2 - 25 = 0$ i.e. $x = -5$ or $x = 5$.

The critical numbers are $x = -5$, $x = 0$, $x = 5$.

10. Find all the relative extrema of the function, and classify each of them as a minimum or a maximum.

$$f(x) = x^4 - 4x^3 + 4x^2$$

$$\begin{aligned} f'(x) &= 4x^3 - 12x^2 + 8x \\ &= 4x(x^2 - 3x + 2) \\ &= 4x(x-1)(x-2) \end{aligned}$$

Critical numbers? f' is never undefined.

f' is zero if $x = 0$ or $x = 1$ or $x = 2$.

		0		1		2		
	$- - - -$	0	$+ + + +$	$+$	$+ + + +$	$+$	$+ + +$	x
	$- - - -$	$-$	$- - - -$	0	$+ + + +$	$+$	$+ + +$	$x-1$
	$- - - -$	$-$	$- - - -$	$-$	$- - - -$	0	$+ + + +$	$x-2$

f' = neg·neg·neg	f' = pos·neg·neg	f' = pos·pos·neg	f' = pos·pos·pos
f' neg	f' pos	f' neg	f' pos
f decreasing	f increasing	f decreasing	f increasing

relative MIN   relative MAX   relative MIN

11. Find all the relative extrema of the function, and classify each of them as a minimum or a maximum.

$$f(x) = 2x + 3x^{2/3}$$

$$f'(x) = 2\cdot 1 + 3\cdot\frac{2}{3}x^{-1/3}$$

$$= 2 + 2x^{-1/3}$$

$$= 2 + \frac{2}{x^{1/3}} = \frac{2x^{1/3}}{x^{1/3}} + \frac{2}{x^{1/3}}$$

$$= \frac{2x^{1/3} + 2}{x^{1/3}}$$

Critical numbers? f' is undefined when $x^{1/3} = 0$

i.e. when $x = 0$

f' is zero when $2x^{1/3} + 2 = 0$

$x^{1/3} + 1 = 0$

$x^{1/3} = -1$

$x = -1$

-1 $\quad$ 0

$- - - - - 0 + + + + + + + +$ $\quad$ $2x^{1/3}+2 = 2(x^{1/3}+1)$

$- - - - - - - - - - 0 + + + +$ $\quad$ $x^{1/3}$

$f' = \text{neg}\cdot\text{neg}$ $\quad$ $f' = \text{pos}\cdot\text{neg}$ $\quad$ $f' = \text{pos}\cdot\text{pos}$

f' pos $\quad$ f' neg $\quad$ f' pos

Minimum at $x = 0$

Maximum at $x = -1$

12. Find all the relative extrema of the function, and classify each of them as a minimum or a maximum.

$$f(x) = \frac{x^2}{x^4+16}$$

$$f'(x) = \frac{(x^2)'(x^4+16) - x^2(x^4+16)'}{(x^4+16)^2}$$

$$= \frac{2x(x^4+16) - x^2 \cdot 4x^3}{(x^4+16)^2} = \frac{2x^5+32x-4x^5}{(x^4+16)^2}$$

$$= \frac{32x-2x^5}{(x^4+16)^2} = \frac{2x(16-x^4)}{(x^4+16)^2}$$

Critical numbers?

Denominator is never 0.

Numerator is 0 if $x=0$ or $x^4=16$

$\hookrightarrow x=-2$ or 2

$$f'(x) = \frac{2x(4+x^2)(4-x^2)}{(x^4+16)^2} = \frac{2x(4+x^2)(2+x)(2-x)}{(x^4+16)^2}$$

The factors $4+x^2$ and $(x^4+16)^2$ are always positive.

		-2		0		2	
$2x$	$- - -$	$-$	$- - -$	0	$+ + +$	$+$	$+ + +$
$2+x$	$- - -$	0	$+ + +$	$+$	$+ + +$	$+$	$+ + +$
$2-x$	$+ + +$	$+$	$+ + +$	$+$	$+ + +$	0	$- - -$
	$f'>0$		$f'<0$		$f'>0$		$f'<0$

Max at $x=-2$

Min at $x=0$

Max at $x=2$

13. Find all the relative extrema of the function, and classify each of them as a minimum or a maximum.

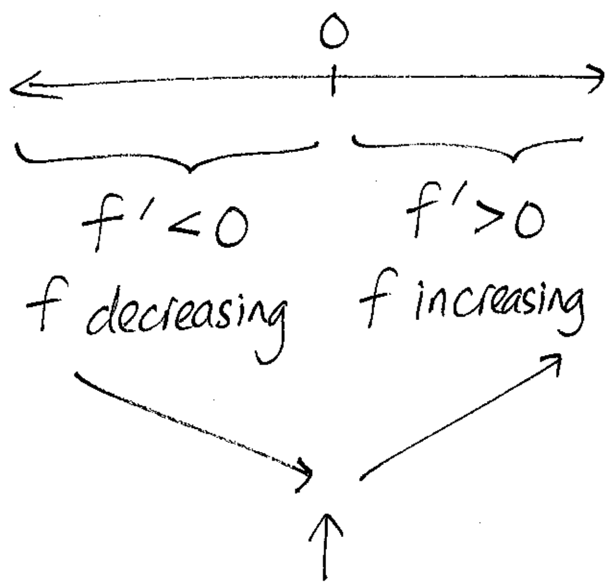
$$f(x) = \ln(2 + x^2)$$

$$f'(x) = \frac{1}{2+x^2} \cdot (2+x^2)' = \frac{1}{2+x^2} \cdot 2x$$

$$f'(x) = \frac{2x}{2+x^2}$$

Top changes sign at $x=0$

Bottom is always positive



Minimum at $x=0$

14. Draw a graph of the function. Label all the critical points, inflection points, and asymptotes.

$$f(x) = \frac{x-3}{4-x}$$

$$f'(x) = \frac{(x-3)'(4-x) - (x-3)(4-x)'}{(4-x)^2}$$

$$= \frac{1(4-x) - (x-3)\cdot(-1)}{(4-x)^2} = \frac{4-x+x-3}{(4-x)^2} = \frac{7}{(4-x)^2}$$

$$f'(x) = 7(4-x)^{-2} \Rightarrow f''(x) = 7\cdot(-2)(4-x)^{-3}\cdot(-1)$$

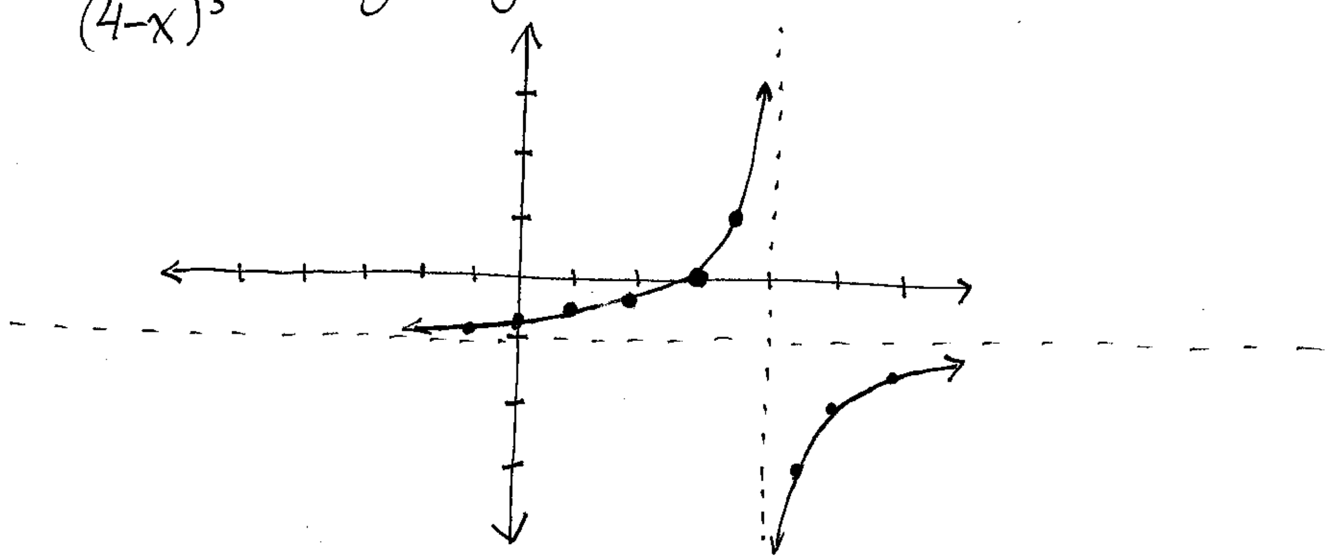
$$= 14(4-x)^{-3} = \frac{14}{(4-x)^3}$$

Vertical asymptote at $x=4$

Horizontal asymptote at $x=-1$

$f'(x) = \frac{7}{(4-x)^2}$ is always positive when it's defined

$f''(x) = \frac{14}{(4-x)^3}$ changes sign at $x=4$



Vertical asymptotes: $x=2,\ x=-2$
Horizontal asymptotes: $y=0$

15. Draw a graph of the function. Label all the critical points, inflection points, and asymptotes.

$$f(x) = \frac{x}{x^2-4}$$

$$f'(x) = \frac{(x)'(x^2-4) - x(x^2-4)'}{(x^2-4)^2} = \frac{1(x^2-4) - x\cdot 2x}{(x^2-4)^2}$$

$$= \frac{x^2-4-2x^2}{(x^2-4)^2} = \frac{-x^2-4}{(x^2-4)^2} = \frac{-(x^2+4)}{(x^2-4)^2}$$

Notice $x^2+4>0$
$(x^2-4)^2$ is never negative because of the square

$$f''(x) = -\frac{(x^2+4)'(x^2-4)^2 - (x^2+4)((x^2-4)^2)'}{((x^2-4)^2)^2}$$

$$= -\frac{2x(x^2-4)^2 - (x^2+4)\cdot 2(x^2-4)\cdot 2x}{(x^2-4)^4}$$

$$= -\frac{2x(x^2-4)[(x^2-4) - (x^2+4)\cdot 2]}{(x^2-4)^4}$$

$$= -\frac{2x(x^2-4)(x^2-4-2x^2-8)}{(x^2-4)^4} = \frac{-2x(-x^2-12)}{(x^2-4)^3}$$

$$= \frac{2x(x^2+12)}{(x^2-4)^3}$$

Changes sign at $x=0,\ x=-2,\ x=2$

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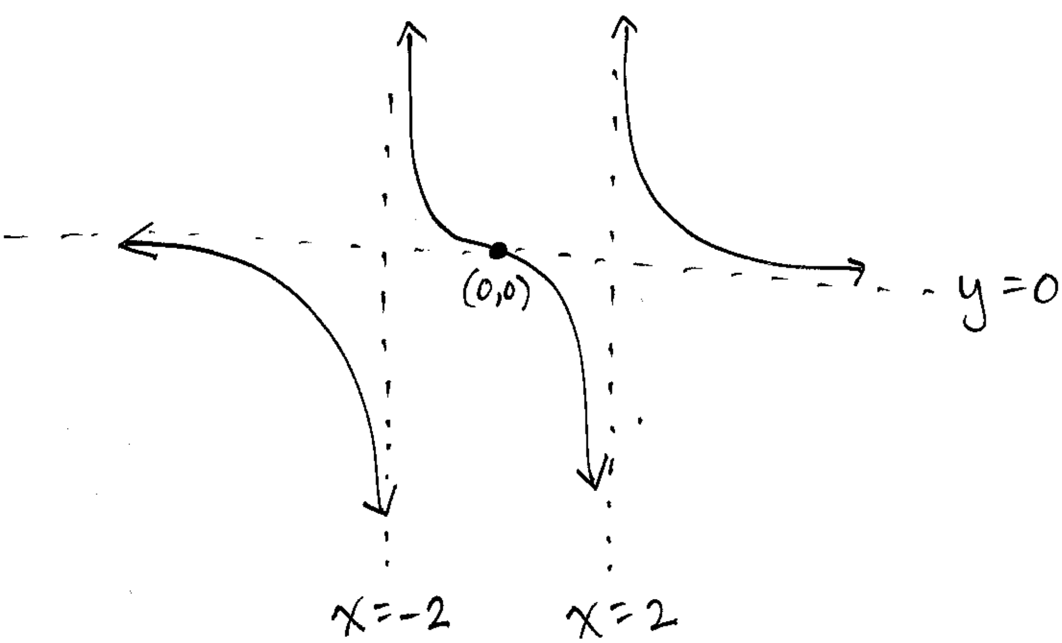
$$f'(x) = \frac{-(x^2+4)}{(x^2-4)^2}$$ Always negative
Undefined at $x=-2$ and $x=2$

$$f''(x) = \frac{2x(x^2+12)}{(x^2-4)^3}$$ Changes sign at $x=-2$
$x=0$
$x=2$

	$x<-2$	$-2<x<0$	$0<x<2$	$x>2$
$2x$	$-$	$-$ (0 at $x=0$)	$+$	$+$
x^2+12	$+$	$+$	$+$	$+$
$(x^2-4)^3$	$+$ (0 at $x=-2$)	$-$	$-$ (0 at $x=2$)	$+$
	$f''<0$	$f''>0$	$f''<0$	$f''>0$



16. Find the absolute maximum and absolute minimum of the function

$$f(x) = (x^2 + x)^{2/3}$$

on the interval $[-2, 3]$.

$$f'(x) = \frac{2}{3}(x^2+x)^{-1/3} \cdot (x^2+x)'$$

$$= \frac{2}{3}(x^2+x)^{-1/3} \cdot (2x+1) = \frac{2(2x+1)}{3(x^2+x)^{1/3}}$$

Critical numbers? Top $= 0 \Rightarrow 2x+1=0 \Rightarrow x = -\frac{1}{2}$

Bottom $= 0 \Rightarrow x^2+x=0 \Rightarrow x=0$ or $x=-1$

$x(x+1)=0$

So we must check the critical numbers in the interval and the endpoints of the interval.

$$f(-2) = ((-2)^2+(-2))^{2/3} = (4-2)^{2/3} = 2^{2/3}$$

$$f(-1) = ((-1)^2+(-1))^{2/3} = (1-1)^{2/3} = 0$$

$$f(-\tfrac{1}{2}) = ((-\tfrac{1}{2})^2+(-\tfrac{1}{2}))^{2/3} = (\tfrac{1}{4}-\tfrac{1}{2})^{2/3} = (-\tfrac{1}{4})^{2/3}$$

$$f(0) = (0^2+0)^{2/3} = 0$$

$$f(3) = (3^2+3)^{2/3} = (9+3)^{2/3} = 12^{2/3}$$

The absolute max is $12^{2/3}$ and the absolute min is $\left(-\frac{1}{4}\right)^{2/3}$

17. Find the absolute maximum and absolute minimum of the function

$$f(x) = \frac{x-2}{x+1}$$

on the interval $(-1, 5]$.

$$f'(x) = \frac{(x-2)'(x+1) - (x-2)(x+1)'}{(x+1)^2}$$

$$= \frac{1(x+1) - (x-2)\cdot 1}{(x+1)^2} = \frac{x+1-x+2}{(x+1)^2} = \frac{3}{(x+1)^2}$$

The only x value that makes f' zero or undefined is $x = -1$

Check x near -1 (technically x cannot be -1)

and $x = 5$

$$f(-0.999) = \frac{-0.999-2}{-0.999+1} = \frac{-2.999}{0.001} = \text{extreme negative number}$$

$$f(5) = \frac{5-2}{5+1} = \frac{3}{6} = \frac{1}{2}$$

The absolute maximum on the given interval is $\frac{1}{2}$.

There is no absolute minimum.

18. Find the absolute maximum and absolute minimum of the function

$$f(x) = \frac{\ln x}{x}$$

on the interval $[1, e^2]$.

$$f'(x) = \frac{(\ln x)' x - (\ln x)(x)'}{x^2}$$

$$= \frac{\frac{1}{x} \cdot x - (\ln x) \cdot 1}{x^2} = \frac{1 - \ln x}{x^2}$$

Critical numbers? $1 - \ln x = 0 \Rightarrow 1 = \ln x \Rightarrow x = e^1 = e$

$x^2 = 0 \Rightarrow x = 0$

Only one critical number in the interval $[1, e^2]$

Check $x = 1$, $x = e$, $x = e^2$

$$f(1) = \frac{\ln 1}{1} = \frac{0}{1} = 0$$

$$f(e) = \frac{\ln e}{e} = \frac{1}{e} \quad \left(\approx \frac{1}{2.7} \approx 0.37\right)$$

$$f(e^2) = \frac{\ln(e^2)}{e^2} = \frac{2}{e^2} = \frac{2}{e} \cdot \frac{1}{e}$$ which is SMALLER than $\frac{1}{e}$

19. Find a number in the closed interval $[\frac{1}{2}, \frac{3}{2}]$ such that the sum of the number and its reciprocal is
a. as small as possible
b. as large as possible.

Let the number be x. Then $\frac{1}{2} \le x \le \frac{3}{2}$.

"Sum of the number and its reciprocal" $= x + \frac{1}{x}$

$f(x) = x + \frac{1}{x} = x + x^{-1}$. (a) minimize $f(x)$
(b) maximize $f(x)$

To minimize or maximize $f(x)$, check critical points and endpoints of domain. $f'(x) = 1 - 1x^{-2} = 1 - \frac{1}{x^2}$

$f' = 0 \Rightarrow 1 - \frac{1}{x^2} = 0 \Rightarrow x^2 = 1 \Rightarrow x = 1$ or -1

Also f' is undefined if $x = 0$. The only critical number in the domain is $x = 1$. Check $x = \frac{1}{2}$, $x = 1$, $x = \frac{3}{2}$.

$f(\frac{1}{2}) = \frac{1}{2} + \frac{2}{1} = \frac{1}{2} + \frac{4}{2} = \frac{5}{2} = 2 + \frac{1}{2}$ or 2.5

$f(1) = 1 + 1 = 2$

$f(\frac{3}{2}) = \frac{3}{2} + \frac{2}{3} = \frac{9}{6} + \frac{4}{6} = \frac{13}{6} = 2 + \frac{1}{6}$

(a) To minimize f, we should choose $x = 1$
(b) To maximize f, we should choose $x = \frac{1}{2}$

20. How should two nonnegative numbers be chosen so that their sum is 1 and the sum of their squares is

a. as large as possible

b. as small as possible?

Let one of the numbers be x. If their sum is 1, then the other number is $1-x$.

We want to maximize or minimize the sum of squares i.e. maximize or minimize $f(x) = x^2 + (1-x)^2$.

Both numbers must be nonnegative. So $x \geq 0$ and $1-x \geq 0$ i.e. $x \leq 1$. Check crit #s and endpoints.

$$f(x) = x^2 + 1 - 2x + x^2 = 1 - 2x + 2x^2$$

$$f'(x) = -2 + 4x. \quad f' = 0 \Rightarrow -2 + 4x = 0$$

$$4x = 2$$

$$x = 1/2$$

Check $x = 0$, $x = \frac{1}{2}$, $x = 1$.

$$f(0) = 0^2 + (1-0)^2 = 0 + 1 = 1$$

$$f(\tfrac{1}{2}) = (\tfrac{1}{2})^2 + (1-\tfrac{1}{2})^2 = (\tfrac{1}{2})^2 + (\tfrac{1}{2})^2 = \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}$$

$$f(1) = 1^2 + (1-1)^2 = 1 + 0 = 1$$

(a) To maximize sum of squares, two numbers should be 0 and 1

(b) To minimize sum of squares, two numbers should be $\frac{1}{2}$ and $\frac{1}{2}$