Tips for remembering the linearization formula:

(change in f) $\approx$ (derivative) $\cdot$ (change in x)

$$f(x) - f(a) \approx f'(a) \cdot (x-a)$$

$$f(x) \approx f(a) + f'(a) \cdot (x-a)$$

MAC2241 Fall 2017
Suggested problems for final exam.
The final exam is **cumulative**.
You should **also** practice the suggested problems for Tests 1 through 3.

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1. Find the linearization of the function $f(x) = \ln x$ at $a = 1$.

Linearization is $f(a) + f'(a) \cdot (x-a)$

$$f'(x) = \frac{1}{x} \qquad f(a) = f(1) = \ln 1 = 0$$

$$f'(a) = f'(1) = \frac{1}{1} = 1$$

Linearization is: $f(a) + f'(a) \cdot (x-a)$

$$= 0 + 1 \cdot (x-1) = \underline{x-1}$$

2. Find the linearization of the function $f(x) = x^{3/4}$ at $a = 16$.

$$f'(x) = \frac{3}{4} x^{-1/4} = \frac{3}{4x^{1/4}}$$

$$f(a) = f(16) = 16^{3/4} = (16^{1/4})^3 = 2^3 = 8$$

$$f'(a) = f'(16) = \frac{3}{4 \cdot 16^{1/4}} = \frac{3}{4 \cdot 2} = \frac{3}{8}$$

Linearization: $f(a) + f'(a) \cdot (x-a)$

$$= 8 + \frac{3}{8} \cdot (x-16)$$

3. Find the linearization of the function $f(x) = \sqrt{1-x}$ at $a = 0$ and use it to approximate the numbers $\sqrt{0.9}$ and $\sqrt{0.99}$.

$$f(x) = (1-x)^{1/2} \quad f'(x) = \frac{1}{2}(1-x)^{-1/2}\cdot(-1) = \frac{-1}{2\sqrt{1-x}}$$

$$f(a) = f(0) = (1-0)^{1/2} = 1 \quad f'(a) = f'(0) = \frac{-1}{2\sqrt{1-0}} = -\frac{1}{2}$$

Linearization: $f(a) + f'(a)\cdot(x-a) = 1 - \frac{1}{2}(x-0) = 1 - \frac{x}{2}$

$$\sqrt{0.9} = \sqrt{1-0.1} = f(0.1) \approx 1 - \frac{0.1}{2} = 1 - 0.05 = 0.95$$

$$\sqrt{0.99} = \sqrt{1-0.01} = f(0.01) \approx 1 - \frac{0.01}{2} = 1 - 0.005 = 0.995$$

4. Find the linearization of the function $f(x) = (1+x)^{1/3}$ at $a = 0$ and use it to approximate the numbers $0.95^{1/3}$ and $1.1^{1/3}$.

$$f'(x) = \frac{1}{3}(1+x)^{-2/3} \quad f(a) = f(0) = (1+0)^{1/3} = 1$$

$$f'(a) = f'(0) = \frac{1}{3}(1+0)^{-2/3} = \frac{1}{3}$$

Linearization: $f(a) + f'(a)\cdot(x-a) = 1 + \frac{1}{3}(x-0) = 1 + \frac{x}{3}$

$$0.95^{1/3} = (1-0.05)^{1/3} = f(-0.05) \approx 1 - \frac{0.05}{3}$$

$$= 1 - 0.016666\ldots$$

$$= 0.983333\ldots$$

$$1.1^{1/3} = (1+0.1)^{1/3} = f(0.1) \approx 1 + \frac{0.1}{3} = 1 + 0.03333\ldots$$

$$= 1.03333\ldots$$

5. Use a linear approximation to estimate the number 2.001^5.

Notice that this is of the form $(2+\text{small number})^5$.

We could choose $f(x)=(2+x)^5$ and $a=0$.

Then $f'(x)=5(2+x)^4\cdot(2+x)'=5(2+x)^4$

$f(a)=f(0)=(2+0)^5=2^5=32$

$f'(a)=f'(0)=5(2+0)^4=5\cdot2^4=5\cdot16=80$

$$f(x)\approx f(a)+f'(a)\cdot(x-a)$$
$$=32+80\cdot(x-0)=32+80x$$

Then $2.001^5=(2+0.001)^5=f(0.001)$

$$\approx 32+80(0.001)=32+0.08=\underline{32.08}$$

Alternative method: Choose $f(x)=x^5$ and $a=2$.

Then $f'(x)=5x^4$. $f(a)=f(2)=2^5=32$

$f'(a)=f'(2)=5\cdot2^4=5\cdot16=80$

$f(x)\approx f(a)+f'(a)\cdot(x-a)=32+80(x-2)$

Then $2.001^5=f(2.001)\approx 32+80(2.001-2)=32+80(0.001)$
$=32.08$

6. Find the critical numbers of the function.

$$f(x) = 4 + \frac{1}{3}x - \frac{1}{2}x^2$$

$f'(x) = 0 + \frac{1}{3} - \frac{1}{2} \cdot 2x = \frac{1}{3} - x$

f' undefined? Never.

$f' = 0$? When $\frac{1}{3} - x = 0 \Rightarrow x = \frac{1}{3}$

The only critical number is $x = \frac{1}{3}$

7. Find the critical numbers of the function.

$$f(x) = x^{3/4} - 2x^{1/4}$$

$f'(x) = \frac{3}{4}x^{-1/4} - 2 \cdot \frac{1}{4}x^{-3/4}$

$= \frac{3}{4x^{1/4}} - \frac{1}{2x^{3/4}} = \frac{3x^{2/4}}{4x^{3/4}} - \frac{2}{4x^{3/4}}$

$= \frac{3x^{1/2} - 2}{4x^{3/4}}.$

f' undefined? When $x^{3/4} = 0 \Rightarrow x = 0$

$f' = 0$? When $3x^{1/2} - 2 = 0$

$3x^{1/2} = 2$

$x^{1/2} = \frac{2}{3}$

$x = \frac{4}{9}$

The critical numbers are 0 and $\frac{4}{9}$

8. Find the absolute maximum and absolute minimum of the function $f(x) = 12 + 4x - x^2$ on the interval $[0, 5]$.

$f'(x) = 0 + 4 - 2x$. Crit pts? f' never undefined. $f' = 0$ if $\underline{x=2}$.

Check critical points and <u>ends</u> of domain: $x = 0$, $x = 2$, $x = 5$.

$f(0) = 12 + 4 \cdot 0 - 0^2 = 12 + 0 - 0 = 12$

$f(2) = 12 + 4 \cdot 2 - 2^2 = 12 + 8 - 4 = 16$

$f(5) = 12 + 4 \cdot 5 - 5^2 = 12 + 20 - 25 = 7$

The absolute max is 16 (at $x=2$) and the absolute min is 7 (at $x=5$)

9. Find the absolute maximum and absolute minimum of the function $f(x) = (x^2 - 1)^3$ on the interval $[-1, 2]$.

$f'(x) = 3(x^2-1)^2 \cdot (x^2-1)' = 3(x^2-1)^2 \cdot 2x$

Critical numbers? f' undefined? Never. $f' = 0$? If $x^2 - 1 = 0$ or $x = 0$

$\Rightarrow x = -1, 0,$ or 1. Check those points and endpoints. $x=-1$ already listed

$x=2$

$f(-1) = ((-1)^2 - 1)^3 = (1-1)^3 = 0^3 = 0$

$f(0) = (0^2 - 1)^3 = (0-1)^3 = (-1)^3 = -1$

$f(1) = (1^2 - 1)^3 = (1-1)^3 = 0^3 = 0$

$f(2) = (2^2 - 1)^3 = (4-1)^3 = 3^3 = 27$

The absolute max is 27 (at $x=2$)

The absolute min is -1 (at $x=0$)

NOTE

10. Find the absolute maximum and absolute minimum of the function $f(x) = \frac{x^2-4}{x^2+4}$ on the interval $[-4, 4]$.

$$f'(x) = \frac{(x^2-4)'(x^2+4) - (x^2-4)(x^2+4)'}{(x^2+4)^2} = \frac{2x(x^2+4) - (x^2-4)2x}{(x^2+4)^2}$$

NOTE

$$= \frac{2x^3+8x-2x^3+8x}{(x^2+4)^2} = \frac{16x}{(x^2+4)^2}.$$

f' undefined? Never. Bottom is never 0.

$f'=0$? When $16x=0$ i.e. $x=0$.

Check crit numbers and endpoints: $x=-4$, $x=0$, $x=4$

$f(-4) = \frac{16-4}{16+4} = \frac{12}{20} = \frac{3}{5}$ $f(0) = \frac{0-4}{0+4} = \frac{-4}{4} = -1$ $f(4) = \frac{16-4}{16+4} = \frac{12}{20} = \frac{3}{5}$

Absolute max is $3/5$, absolute min is -1.

11. Find the absolute maximum and absolute minimum of the function $f(x) = \ln(x^2 + 2x + 2)$ on the interval $[-2, 0]$.

$$f'(x) = \frac{1}{x^2+2x+2} \cdot (x^2+2x+2)' = \frac{2x+2}{x^2+2x+2}$$

f' undefined? Denominator $=0$? Never, because x^2+2x+2
$= x^2+2x+1+1$
$= (x+1)^2+1.$

$f'=0$? When $2x+2=0$, i.e. $x=-1$.

Check critical points and endpoints: $x=-2$, $x=-1$, $x=0$.

$f(-2) = \ln((-2)^2+2(-2)+2) = \ln(4-4+2) = \ln 2$

$f(-1) = \ln((-1)^2+2(-1)+2) = \ln(1-2+2) = \ln 1 = 0$

$f(0) = \ln(0^2+2(0)+2) = \ln(0+0+2) = \ln 2$

Absolute max is $\ln 2$ and absolute min is 0.

12. Find the intervals on which f is increasing, decreasing, concave up, and concave down.

$$f(x) = 2x^3 + 3x^2 - 36x$$

$$f'(x) = 2\cdot 3x^2 + 3\cdot 2x - 36\cdot 1$$
$$= 6x^2 + 6x - 36 = 6(x^2 + x - 6)$$
$$= 6(x+3)(x-2)$$

$$f''(x) = 6\cdot 2x + 6\cdot 1 = 0 = 12x + 6$$

f' undefined? Never. $f'=0$? When $x=-3$ or $x=2$.

$x=-3$ $\quad$ $x=2$

$\underbrace{\quad\quad}_{f' \text{ is pos}}$ $\underbrace{\quad\quad}_{f' \text{ is neg}}$ $\underbrace{\quad\quad}_{f' \text{ is pos}}$

f'' undefined? Never. $f''=0$? When
$$12x + 6 = 0$$
$$12x = -6$$
$$x = \frac{-6}{12} = \frac{-1}{2}$$

$x = -1/2$

$\underbrace{\quad\quad}_{f'' \text{ is neg}}$ $\underbrace{\quad\quad}_{f'' \text{ is pos}}$

-3 $\quad$ 2

f incr $\quad$ f decr $\quad$ f incr

$-1/2$

f conc down $\quad$ f conc up

Could combine if we want

-3 $\quad$ $-1/2$ $\quad$ 2

f incr	f decr	f decr	f incr
conc down	conc down	conc up	conc up

13. Find the intervals on which f is increasing, decreasing, concave up, and concave down.

$$f(x) = \frac{x^2}{x^2+3}$$

$$f'(x) = \frac{(2x)(x^2+3)-(x^2)(2x)}{(x^2+3)^2} = \frac{2x^3+6x-2x^3}{(x^2+3)^2} = \frac{6x}{(x^2+3)^2}$$

$$f''(x) = \frac{(6x)'(x^2+3)^2 - 6x\left((x^2+3)^2\right)'}{(x^2+3)^4}$$

$$= \frac{6(x^2+3)^2 - 6x\cdot \overbrace{2(x^2+3)\cdot 2x}^{\text{chain rule}}}{(x^2+3)^4} = \frac{6(x^2+3)\left[(x^2+3) - \overbrace{x\cdot 2\cdot 2x}^{4x^2}\right]}{(x^2+3)^4}$$

$$= \frac{6\left[x^2+3-4x^2\right]}{(x^2+3)^3} = \frac{6(3-3x^2)}{(x^2+3)^3} = \frac{18(1-x^2)}{(x^2+3)^3}$$

Notice that the denominators $(x^2+3)^2$ and $(x^2+3)^3$ are never 0, (always positive) so f' and f'' are never undefined. When is $f'=0$? When $x=0$. When is $f''=0$? When $x^2=1$ i.e. $x=-1$ or 1.

$x=0$

f' neg	f' pos
f decr	f incr

$x=-1$ $\quad$ $x=1$

f'' neg	f'' pos	f'' neg
f conc down	f conc up	f conc down

14. Find the maximum value of the function $f(x) = x + \sqrt{1-x}$ on its domain.

What is the domain? Function contains $\sqrt{1-x}$, so we must have $1-x \geq 0$, i.e. $x \leq 1$.

The "ends" of the domain are $-\infty$ and 1.

Now, $f(x) = x + (1-x)^{1/2}$

so $f'(x) = 1 + \frac{1}{2}(1-x)^{-1/2} \cdot (-1) = 1 - \frac{1}{2\sqrt{1-x}}$

$= \frac{2\sqrt{1-x}}{2\sqrt{1-x}} - \frac{1}{2\sqrt{1-x}} = \frac{2\sqrt{1-x} - 1}{2\sqrt{1-x}}$

Critical points? f' undefined? When $2\sqrt{1-x} = 0$ i.e. $x=1$

$f'=0$? When $2\sqrt{1-x} - 1 = 0 \Rightarrow 2\sqrt{1-x} = 1$

$\Rightarrow \sqrt{1-x} = \frac{1}{2} \Rightarrow 1-x = \frac{1}{4} \Rightarrow x = \frac{3}{4}$.

So, check x near $-\infty$, $x = \frac{3}{4}$, and $x = 1$.

$f\left(\frac{3}{4}\right) = \frac{3}{4} + \sqrt{1 - \frac{3}{4}} = \frac{3}{4} + \sqrt{\frac{1}{4}} = \frac{3}{4} + \frac{1}{2} = \frac{3}{4} + \frac{2}{4} = \frac{5}{4}$

$f(1) = 1 + \sqrt{1-1} = 1 + 0 = 1$

$f(-1000000) = -1000000 + \sqrt{1000001} \approx -1000000 + 1000$

Extreme negative number

The maximum value is $\frac{5}{4}$.

15. Find the intervals where $f(x)$ is increasing, decreasing, concave up, and concave down.

$$f(x) = 2 + 2x^2 - x^4$$

$$f'(x) = 4x - 4x^3$$

$$f''(x) = 4 - 12x^2$$

$$f'(x) = 4x(1-x^2) = 4x(1+x)(1-x)$$

$$f''(x) = 4(1-3x^2)$$ which can be factored as $4(1+\sqrt{3}x)(1-\sqrt{3}x)$

When might f' change sign? When $x = 0, -1,$ or 1.

When might f'' change sign? When $3x^2 = 1$

$x^2 = 1/3 \Rightarrow x = \pm \frac{1}{\sqrt{3}}$

		$x=-1$		$x=0$		$x=1$	
$4x$	− − −	−	− − −	0	+ + +	+	+ + +
$1+x$	− − −	0	+ + +	+	+ + +	+	+ + +
$1-x$	+ + +	+	+ + +	+	+ + +	0	− − −
f' $4x(1+x)(1-x)$	f' pos f incr		f' neg f decr		f' pos f incr		f' neg f decr

	$x=\frac{-1}{\sqrt{3}}$		$x=\frac{1}{\sqrt{3}}$	
f'' neg f conc DOWN		f'' pos f conc UP		f'' neg f conc DOWN

16. Find the intervals where $f(x)$ is increasing, decreasing, concave up, and concave down.

$$f(x) = (x+1)^5 - 5x - 2$$

$$f'(x) = 5(x+1)^4 \cdot 1 - 5$$

$$f''(x) = 20(x+1)^3$$

f' undefined? Never. $f'=0$?

$$5(x+1)^4 = 5$$
$$(x+1)^4 = 1$$
$$x+1 = 1 \text{ or } -1$$
$$x = 0 \text{ or } -2$$

f'' undefined? Never. $f''=0$?

$$20(x+1)^3 = 0$$
$$(x+1)^3 = 0$$
$$x+1 = 0$$
$$x = -1$$

$x=-2$ $\quad$ $x=0$

try e.g. $x=-1000$	e.g. $x=-1$	e.g. $x=1000$
f' pos	f' neg	f' pos
f increasing	f decreasing	f increasing

$x=-1$

f'' negative	f'' positive
f concave DOWN	f concave UP

17. Find the intervals where $f(x)$ is increasing and decreasing.

$$f(x) = x\sqrt{x+3}$$

First note that formula contains $\sqrt{x+3}$ so we must have $x+3 \geq 0$

$$x \geq -3$$

Now: $f(x) = x(x+3)^{1/2}$

$$f'(x) = (x)'(x+3)^{1/2} + x\left((x+3)^{1/2}\right)'$$

$$= 1(x+3)^{1/2} + x \cdot \frac{1}{2}(x+3)^{-1/2} \cdot 1$$

$$= \sqrt{x+3} + \frac{x}{2\sqrt{x+3}} = \frac{2(\sqrt{x+3})^2}{2\sqrt{x+3}} + \frac{x}{2\sqrt{x+3}}$$

$$= \frac{2(x+3)+x}{2\sqrt{x+3}} = \frac{2x+6+x}{2\sqrt{x+3}} = \frac{3x+6}{2\sqrt{x+3}}$$

f' undefined? When $2\sqrt{x+3} = 0$ i.e. $x = -3$.

$f' = 0$? When $3x+6 = 0$ i.e. $x = -2$.

$x=-3$ $\qquad\qquad$ $x=-2$

$3x+6 = 3(x+2)$: $- - - - - - - - - 0 + + + + + + + + +$

$2\sqrt{x+3}$: $+ + + + + + + + + + + + + + + + + + +$

$f' = \frac{3x+6}{2\sqrt{x+3}}$: $- - - - - - 0 + + + + + +$

f' neg $\qquad$ f' pos

f decreasing $\qquad$ f increasing

18. Find the intervals where $f(x)$ is increasing, decreasing, concave up, and concave down.

$$f(x) = \ln(x^4 + 27)$$

$f = \ln u$ and $u = x^4 + 27$

Chain rule

$$f'(x) = \frac{1}{x^4+27} \cdot (x^4+27)' = \frac{1}{x^4+27} \cdot 4x^3 = \frac{4x^3}{x^4+27}$$

Bottom is always positive

Top has same sign as x

$$f''(x) = \frac{(4x^3)'(x^4+27) - (4x^3)(x^4+27)'}{(x^4+27)^2}$$

$$= \frac{12x^2(x^4+27) - 4x^3 \cdot 4x^3}{(x^4+27)^2} = \frac{12x^6 + 12 \cdot 27x^2 - 16x^6}{(x^4+27)^2}$$

$$= \frac{12 \cdot 27x^2 - 4x^6}{(x^4+27)^2} = \frac{4x^2(3 \cdot 27 - x^4)}{(x^4+27)^2} = \frac{4x^2(81 - x^4)}{(x^4+27)^2}$$

Notice $4x^2$ and $(x^4+27)^2$ are never negative.

$81 - x^4$ can change sign if $x^4 = 81$ i.e. $x = 3$ or -3

$x=0$

$x<0$	$x>0$
f' negative	f' positive
f decreasing	f increasing

$x=-3$ $\quad$ $x=3$

$x<-3$	$-3<x<3$	$x>3$
f'' neg	f'' pos	f'' neg
f conc DOWN	f conc UP	f conc DOWN

19. Find the limit.

$$\lim_{x\to\pi/2^+} \frac{\cos x}{1-\sin x}$$

Notice $\cos\frac{\pi}{2} = 0$ and $1-\sin\frac{\pi}{2} = 1-1 = 0$.

So this limit problem is of the form $\frac{0}{0}$. Use L'Hopital.

$f(x)=\cos x \Rightarrow f'(x) = -\sin x$

$g(x)=1-\sin x \Rightarrow g'(x) = -\cos x$

$$\lim_{x\to\frac{\pi}{2}^+} \frac{-\sin x}{-\cos x}$$

If $x \approx \frac{\pi}{2}$, then $-\sin x \approx -1$. If x slightly more than $\frac{\pi}{2}$, then $-\cos x$ is positive near 0.

$$\frac{\text{Near } -1}{\text{positive near } 0} \longrightarrow \boxed{-\infty.}$$

Can also say limit does not exist.

20. Find the limit.

$$\lim_{x\to 0} \frac{\sin 4x}{\tan 5x}$$

If $x=0$ then $\sin 4x = \sin(4\cdot 0) = \sin 0 = 0$

$\tan 5x = \tan(5\cdot 0) = \tan 0 = 0$

So limit is of the form $\frac{0}{0}$ so can use L'Hopital.

$f(x)=\sin(4x) \Rightarrow f'(x) = \cos(4x)\cdot 4$

$g(x)=\tan(5x) \Rightarrow g'(x) = \sec^2(5x)\cdot 5$

$$\lim_{x\to 0} \frac{\cos(4x)\cdot 4}{\sec^2(5x)\cdot 5} = \frac{\cos(0)\cdot 4}{\sec^2(0)\cdot 5} = \frac{1\cdot 4}{1\cdot 5} = \frac{4}{5}$$

21. Find the limit.

$$\lim_{x\to 0} \frac{e^{3x}-1}{x}$$

If $x=0$ then $e^{3x}-1 = e^{0}-1 = 1-1=0$ so limit has $\frac{0}{0}$ form

$f(x) = e^{3x}-1 \Rightarrow f'(x) = e^{3x}\cdot 3 \quad g(x) = x \Rightarrow g'(x) = 1$

$$\lim_{x\to 0} \frac{e^{3x}\cdot 3}{1} = \frac{e^{0}\cdot 3}{1} = \frac{1\cdot 3}{1} = 3$$

22. Find the limit.

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{\ln x}{x^{1/2}}$$

When $x\to\infty$, both $\ln x$ and $\sqrt{x}$ approach infinity,

so this has the form $\frac{\infty}{\infty}$, so use L'Hopital

$f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}. \quad g(x) = x^{1/2} \Rightarrow g'(x) = \frac{1}{2}x^{-1/2}$

$$\lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty}\left(\frac{1}{x} \div \frac{1}{2}x^{-1/2}\right) = \lim_{x\to\infty}\left(\frac{1}{x} \div \frac{1}{2\sqrt{x}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{1}{x}\cdot\frac{2\sqrt{x}}{1}\right) = \lim_{x\to\infty}\frac{2}{\sqrt{x}} = 0 \quad \left(\text{has the form } \frac{2}{\infty}\right)$$

23. Find the limit.

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x}$$ $\frac{\infty}{\infty}$ form. Can use L'Hopital.

$$= \lim_{x\to\infty} \frac{2(\ln x)\cdot\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{2\ln x}{x}.$$

$\frac{\infty}{\infty}$ again. L'Hopital again.

$$= \lim_{x\to\infty} \frac{\frac{2}{x}}{1} = \lim_{x\to\infty} \frac{2}{x} = 0.$$

24. Find the limit.

$$\lim_{x\to 0} \frac{5^x - 3^x}{x}$$ $$\frac{5^0-3^0}{0} = \frac{1-1}{0} = \frac{0}{0}$$

Indeterminate $\frac{0}{0}$. L'Hopital

$$= \lim_{x\to 0} \frac{5^x \ln 5 - 3^x \ln 3}{1}$$

$$= \frac{5^0 \ln 5 - 3^0 \ln 3}{1} = \frac{1\ln 5 - 1\ln 3}{1}$$

$$= \ln 5 - \ln 3$$

25. Find the limit.

$$\lim_{x\to 0}\frac{e^x-1-x}{x^2}$$

$$\frac{e^0-1-0}{0^2}=\frac{1-1-0}{0}=\frac{0}{0}$$ Indeterminate Use L'Hopital

$$\lim_{x\to 0}\frac{e^x-1-x}{x^2}=\lim_{x\to 0}\frac{e^x-0-1}{2x}$$

Notice $\frac{e^0-0-1}{2\cdot 0}=\frac{1-0-1}{0}=\frac{0}{0}$ again, so use L'Hopital again

$$\lim_{x\to 0}\frac{e^x-1}{2x}=\lim_{x\to 0}\frac{e^x}{2}$$

$$=\frac{e^0}{2}=\frac{1}{2}$$

26. Find two positive numbers whose product is 100 and whose sum is a minimum.

Let the two numbers be x and y.

Since their product is 100, we have $xy = 100$, so $y = \frac{100}{x}$.

We want to minimize the sum $x+y$, which is $x + \frac{100}{x}$.

Let $f(x) = x + \frac{100}{x} = x + 100x^{-1}$.

Then $f'(x) = 1 - 100x^{-2} = 1 - \frac{100}{x^2}$.

f' undefined? When $x=0$. $f'=0$? If $1 - \frac{100}{x^2} = 0$

i.e. $1 = \frac{100}{x^2}$ i.e. $x^2 = 100$ i.e. $x = \pm 10$. But the question said x and y must be positive. So check $x = 10$ and the "ends" of the domain, i.e. x near 0 and x near ∞.

$f(10) = 10 + \frac{100}{10} = 10 + 10 = 20$

$f(\text{near } 0) = \text{near } 0 + \frac{100}{\text{near } 0} = \text{near } 0 + \text{near } \infty = \text{near } \infty$

$f(\text{near } \infty) = \text{near } \infty + \frac{100}{\text{near } \infty} = \text{near } \infty + \text{near } 0 = \text{near } \infty$

The sum is minimized when $x = 10$. The two numbers should be 10 and 10.

27. The sum of two positive numbers is 16. What is the smallest possible value of the sum of their squares?

Let the two numbers be x and y.

Given their sum is 16, so $x+y=16$, so $y=16-x$.

Given the two numbers are positive. So $x \geq 0$ and $x \leq 16$.

Want to minimize the sum of their squares, which is x^2+y^2, which is $x^2+(16-x)^2$.

Let $f(x) = x^2+(16-x)^2 = x^2+16^2-32x+x^2$
$= 2x^2-32x+16^2$. Then $f'(x) = 4x-32$.

f' is never undefined. f' is zero if $x=8$.

Check $x=0$, $x=8$, $x=16$. (Critical points and endpoints.)

$f(0) = 0^2+(16-0)^2 = 16^2$ $(16^2=256)$

$f(8) = 8^2+(16-8)^2 = 8^2+8^2$ definitely smaller than 16^2.

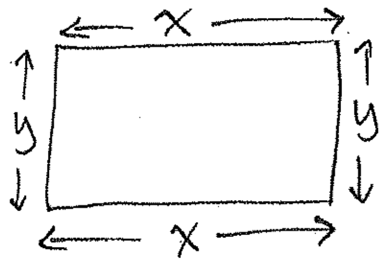
$= 64+64 = 128$

$f(16) = 16^2+(16-16)^2 = 16^2$

The smallest possible value for the sum of the squares is 128.

28. Find the dimensions of a rectangle with perimeter 100 m whose area is as large as possible.

DRAW A PICTURE and MAKE UP NAMES.



GIVEN: perimeter must be 100

i.e. $2x + 2y = 100$

$x + y = 50$

Want to MAXIMIZE the area. Area $= A = xy$

Can we write as a function of one variable?

We know $x+y=50$, so $y = 50 - x$

so $A = xy = x(50-x) = 50x - x^2$

We want to maximize A. If $A = 50x - x^2$ then $A' = 50 - 2x$. When is A' zero or undefined?

A' is never undefined. A' is zero when $x = 25$.

What are the ends of the domain? x is a length, so $x \geq 0$, and also $x \leq 50$ because total perimeter is 100.

$x = 0 \Rightarrow A = 0 \cdot (50-0) = 0$

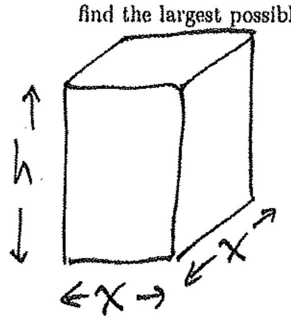
$x = 25 \Rightarrow A = 25 \cdot (50-25) = 25 \cdot 25$ ← Max area. Happens when $x = 25$, $y = 25$.

$x = 50 \Rightarrow A = 50 \cdot (50-50) = 0$

25 m by 25 m

DRAW A PICTURE and MAKE UP NAMES

29. If 1200 cm^2 is available to make a box with a square base and an open top, find the largest possible volume of the box.



"Floor" is x by x (square base)
Four "walls"
Each wall is h by x
No top (box has open top)

GIVEN: Total surface area is 1200

That is, $x^2 + 4xh = 1200$

Want to MAXIMIZE the volume. Volume $= V = x^2 h$.

Can we rewrite volume as a function of one variable?

We know $x^2 + 4xh = 1200 \Rightarrow 4xh = 1200 - x^2$

$\Rightarrow h = \dfrac{1200 - x^2}{4x}$. Then $V = x^2 h = x^2 \cdot \dfrac{1200 - x^2}{4x}$

$= x \cdot \dfrac{1200 - x^2}{4} = 300x - \frac{1}{4}x^3$.

Then $V' = 300 - \frac{3}{4}x^2$. Note V' is never undefined.

V' is zero when $300 = \frac{3}{4}x^2 \Rightarrow 100 = \dfrac{x^2}{4} \Rightarrow 400 = x^2$

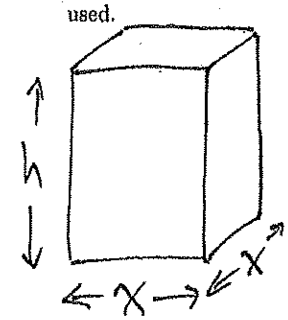
$\Rightarrow x = 20$. (Remember x is a length.) So $x \geq 0$

Upper bound for x? $x^2 \leq 1200$. i.e. $x \leq \sqrt{1200}$.

$x = 0 \Rightarrow V = 0 \qquad x = \sqrt{1200} \Rightarrow V = 0$

$V = \frac{1}{4}x(1200 - x^2) \qquad x = 20 \Rightarrow V = \frac{1}{4} \cdot 20(1200 - 400) = 5 \cdot 800 = 4000$

30. A box with a square base and an open top must have a volume of 32,000 cm^3. Find the dimensions of the box that minimize the amount of material used.



GIVEN: Volume $= 32,000$

i.e. $x^2 h = 32,000$

Want to MINIMIZE surface area

Surface area $S = x^2 + 4xh$ (bottom, four walls, no top)

If $x^2 h = 32,000$ then $h = \frac{32,000}{x^2}$

$$S = x^2 + 4x \cdot \frac{32,000}{x^2} = x^2 + 128,000x^{-1}$$

Then $S' = 2x - 128,000x^{-2} = 2x - \frac{128,000}{x^2}$

S' is undefined when $x=0$. S' is zero when

$$2x = \frac{128,000}{x^2} \Rightarrow 2x^3 = 128,000 \Rightarrow x^3 = 64,000$$

$\Rightarrow x = 40$. Check $x=40$ and "ends" of domain: $x=0$, $x \approx \infty$

$S = x^2 + \frac{128,000}{x}$ $\Rightarrow$ If $x \approx 0$, S is very large

If $x \approx \infty$, S is very large

If $x=40$ then $S = 40^2 + \frac{128,000}{40} = 1600 + 3200$

$= 4800$ is min

Should choose $x=40$ and $h = \frac{32000}{40^2} = 20$

31. Find the antiderivative of the function.

$$f(x) = x^2 - 3$$

Antiderivative is $\frac{x^3}{3} - 3x + C$.

Can also write $F(x) = \frac{x^3}{3} - 3x + C$

or $\int f(x)\,dx = \frac{x^3}{3} - 3x + C$ or $\int (x^2 - 3)\,dx = \frac{x^3}{3} - 3x + C$

32. Find the antiderivative of the function.

$$f(x) = x^{-2}$$

$$\frac{x^{-2+1}}{-2+1} + C$$

$$= \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C$$

33. Find the antiderivative of the function.

$$f(x) = x^4 - \frac{3}{4}x^2 + \frac{2}{3}x - 1$$

$$\frac{x^5}{5} - \frac{3}{4} \cdot \frac{x^3}{3} + \frac{2}{3} \cdot \frac{x^2}{2} - x + C$$

$$= \frac{x^5}{5} - \frac{x^3}{4} + \frac{x^2}{3} - x + C$$

34. Find the antiderivative of the function.

$$f(x) = x^{4/5}$$

$$\frac{x^{\frac{4}{5}+1}}{\frac{4}{5}+1} + C$$

$$= \frac{x^{9/5}}{9/5} + C = \frac{5}{9}x^{9/5} + C$$

35. Find the antiderivative of the function.

$$f(x) = 2x - e^x$$

$$2 \cdot \frac{x^2}{2} - e^x + C$$

$$= x^2 - e^x + C$$

36. Find the antiderivative of the function.

$$f(x) = (1 + 2x)^2$$

$$f(x) = 1 + 4x + 4x^2$$

$$\int f(x)\,dx = x + \frac{4x^2}{2} + \frac{4x^3}{3} + C$$

$$= x + 2x^2 + \frac{4}{3}x^3 + C$$

37. Find the antiderivative of the function.

$$f(x) = \frac{x+1}{\sqrt{x}}$$

$$f(x) = \frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}} = x^{1/2} + x^{-1/2}$$

$$\int f(x)\,dx = \frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + C$$

$$= \frac{2}{3}x^{3/2} + 2x^{1/2} + C$$

38. Find the antiderivative of the function.

$$f(x) = \frac{1}{2x}$$

$$f(x) = \frac{1}{2} \cdot \frac{1}{x}$$

$$\int f(x)\,dx = \int \frac{1}{2} \cdot \frac{1}{x}\,dx = \frac{1}{2}\int \frac{1}{x}\,dx$$

$$= \frac{1}{2}\ln|x| + C$$