WRITE YOUR NAME:

MAC 2241 Section U01 Final Exam
Friday April 27th
Total possible score: 30 points (3 points per page)

Question 1. If $f(x) = \frac{1}{x}$, evaluate the following and simplify.

$$\frac{f(x+h) - f(x)}{h}$$

$$\frac{f(x+h) - f(x)}{h} = \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \cdot \frac{x(x+h)}{x(x+h)}$$

$$= \frac{\frac{x(x+h)}{x+h} - \frac{x(x+h)}{x}}{hx(x+h)} = \frac{x - (x+h)}{hx(x+h)}$$

$$= \frac{x - x - h}{hx(x+h)} = \frac{-h}{hx(x+h)} = \frac{-1}{x(x+h)}$$

Question 2. Solve the equation for x.

$$3 \cdot 2^{x-7} = 24$$

$$\frac{3 \cdot 2^{x-7}}{3} = \frac{24}{3}$$

$$2^{x-7} = 8 \qquad \text{But } 2^3 = 8$$

$$2^{x-7} = 2^3$$

$$x - 7 = 3$$

$$x = 3 + 7$$

$$x = 10$$

Question 3. Evaluate the limit.

$$\lim_{n\to\infty} \frac{12n^3}{\sqrt{9n^6+16n^4}}$$

Answer is $\dfrac{12}{\sqrt{9}} = \dfrac{12}{3} = 4.$

Shortcut: $\displaystyle\lim_{n\to\infty} \frac{12n^3}{\sqrt{9n^6 + (\text{smaller powers})}}$

$$= \lim_{n\to\infty} \frac{12n^3}{\sqrt{9n^6}} = \lim_{n\to\infty} \frac{12n^3}{3n^3} = \frac{12}{3} = 4$$

Longer method: $\displaystyle\lim_{n\to\infty} \frac{12n^3}{\sqrt{9n^6+16n^4}} \cdot \frac{\frac{1}{n^3}}{\frac{1}{n^3}}$

$$= \lim_{n\to\infty} \frac{\frac{12n^3}{n^3}}{\frac{\sqrt{9n^6+16n^4}}{\sqrt{n^6}}} = \lim_{n\to\infty} \frac{12}{\sqrt{\frac{9n^6}{n^6} + \frac{16n^4}{n^6}}}$$

$$= \lim_{n\to\infty} \frac{12}{\sqrt{9+\frac{16}{n^2}}} = \frac{12}{\sqrt{9+0}} = \frac{12}{3} = 4$$

Question 4. Evaluate the limit.

$$\lim_{x \to -3} \frac{\frac{1}{3} + \frac{1}{x}}{3 + x}$$

$$\lim_{x \to -3} \frac{\frac{1}{3} + \frac{1}{x}}{3+x} \cdot \frac{3x}{3x} = \lim_{x \to -3} \frac{\frac{3x}{3} + \frac{3x}{x}}{(3+x) \cdot 3x}$$

$$= \lim_{x \to -3} \frac{x + 3}{(3+x) \cdot 3x} = \lim_{x \to -3} \frac{1}{3x}$$

$$= -\frac{1}{9}$$

Question 5. Evaluate the limit.

$$\lim_{x\to 0}\left(\frac{2}{x}-\frac{6}{3x+x^2}\right)$$

$$\lim_{x\to 0}\left(\frac{2}{x}-\frac{6}{x\cdot(3+x)}\right)=\lim_{x\to 0}\left(\frac{2\cdot(3+x)}{x\cdot(3+x)}-\frac{6}{x\cdot(3+x)}\right)$$

$$=\lim_{x\to 0}\left(\frac{6+2x}{x\cdot(3+x)}-\frac{6}{x\cdot(3+x)}\right)$$

$$=\lim_{x\to 0}\frac{2x}{x\cdot(3+x)}=\lim_{x\to 0}\frac{2}{3+x}$$

$$=\frac{2}{3}$$

$$f(\square) = \frac{1}{\sqrt{\square}}$$

Question 6. Find the derivative of the function using the definition of the derivative.

$$f(x) = \frac{1}{\sqrt{x}}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} \cdot \frac{\sqrt{x}\cdot\sqrt{x+h}}{\sqrt{x}\cdot\sqrt{x+h}}$$

$$= \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\cdot\sqrt{x}\cdot\sqrt{x+h}} \cdot \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}}$$

$$= \lim_{h\to 0} \frac{x - (x+h)}{h\cdot\sqrt{x}\cdot\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)} = \lim_{h\to 0} \frac{-h}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}$$

$$= \lim_{h\to 0} \frac{-1}{\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)} = \frac{-1}{\sqrt{x}\cdot\sqrt{x}\cdot\underbrace{\left(\sqrt{x}+\sqrt{x}\right)}_{2\sqrt{x}}} = \frac{-1}{2x^{3/2}}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Question 7. Find $f'(x)$ and simplify as much as possible.

$$f(x) = \frac{x^3}{1-3x^2}$$

$$f'(x) = \frac{(x^3)'(1-3x^2) - x^3(1-3x^2)'}{(1-3x^2)^2}$$

$$= \frac{3x^2(1-3x^2) - x^3\cdot(-6x)}{(1-3x^2)^2}$$

$$= \frac{3x^2 - 9x^4 + 6x^4}{(1-3x^2)^2} = \frac{3x^2 - 3x^4}{(1-3x^2)^2}$$

or $$\frac{3x^2(1-x^2)}{(1-3x^2)^2}$$

or $$\frac{3x^2(1+x)(1-x)}{(1-3x^2)^2}$$

Question 8. Find $f'(x)$, and find the intervals where $f(x)$ is increasing and decreasing.

$$f(x) = (x^2 - 8x + 15)^{2017}$$

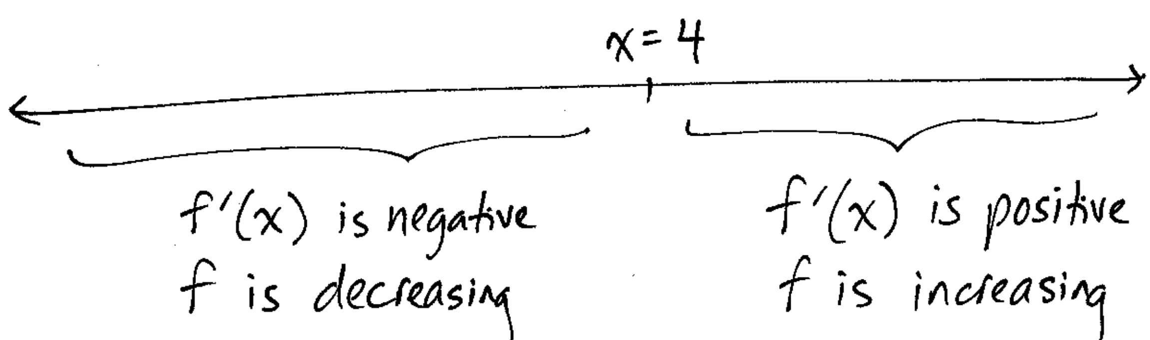
Chain rule

$$f'(x) = 2017\left(x^2-8x+15\right)^{2016} \cdot \left(x^2-8x+15\right)'$$

$$= \underbrace{2017\left(x^2-8x+15\right)^{2016}}_{\substack{\text{Took deriv. of outside} \\ \text{Left inside alone}}} \cdot \underbrace{\left(2x-8\right)}_{\text{Deriv. of inside}}$$

Notice that because of the <u>even</u> exponent, $(x^2-8x+15)^{2016}$ is never negative.

f' will change sign when $2x-8$ changes sign i.e. when $x=4$.



Question 9. Find the critical numbers of the function, and find the intervals where $f(x)$ is increasing or decreasing.

$$f(x) = x^{6/5} - 3x^{1/5}$$

$$f'(x) = \frac{6}{5}x^{1/5} - 3\cdot\frac{1}{5}x^{-4/5}$$

$$= \frac{6}{5}x^{1/5} - \frac{3}{5x^{4/5}}$$

$$= \frac{6x}{5x^{4/5}} - \frac{3}{5x^{4/5}} = \frac{6x-3}{5x^{4/5}}$$

$$= \frac{3(2x-1)}{5(x^{1/5})^4}$$

Critical numbers: $x = \frac{1}{2}$

$x = 0$

		0		1/2	
$2x-1$	$- - - - - -$	$-$	$- - - - -$	0	$+ + + + + +$
$(x^{1/5})^4$	$+ + + + + + +$	0	$+ + + + + + +$	$+$	$+ + + + + +$
	f′ neg f decreasing		f′ neg f decreasing		f′ pos f increasing

Question 10. Find the absolute maximum and the absolute minimum of the function $f(x) = (x^2 - 1)^3$ on the interval $[-1, 2]$.

$$f'(x) = 3(x^2-1)^2 \cdot 2x \qquad \text{(chain rule)}$$

Critical numbers: $x^2 - 1 = 0$ or $x = 0$

$x = \pm 1$

Check $x = -1$, $x = 0$, $x = 1$ and endpoints

$x = -1$ already listed.

Check $x = 2$.

$$f(-1) = ((-1)^2 - 1)^3 = (1-1)^3 = 0$$

$$f(0) = (0^2 - 1)^3 = (0-1)^3 = (-1)^3 = -1$$

$$f(1) = (1^2 - 1)^3 = (1-1)^3 = 0$$

$$f(2) = (2^2 - 1)^3 = (4-1)^3 = 3^3 = 27$$

Abs max is 27 and abs min is -1.