MAC2311 Section U08
Suggested problems for final exam.
The final exam is **cumulative**.
You should **also** practice the
suggested problems for Tests 1 through 3.

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1. Draw a graph of the function. Label all the critical points, inflection points, and asymptotes.

$$f(x) = \frac{x-3}{4-x}$$

Vertical asymptote at $x=4$. Note $f(3.999) = \dfrac{\text{positive near 1}}{\text{positive near 0}}$ = extreme positive

and $f(4.001) = \dfrac{\text{positive near 1}}{\text{negative near 0}}$ = extreme negative.

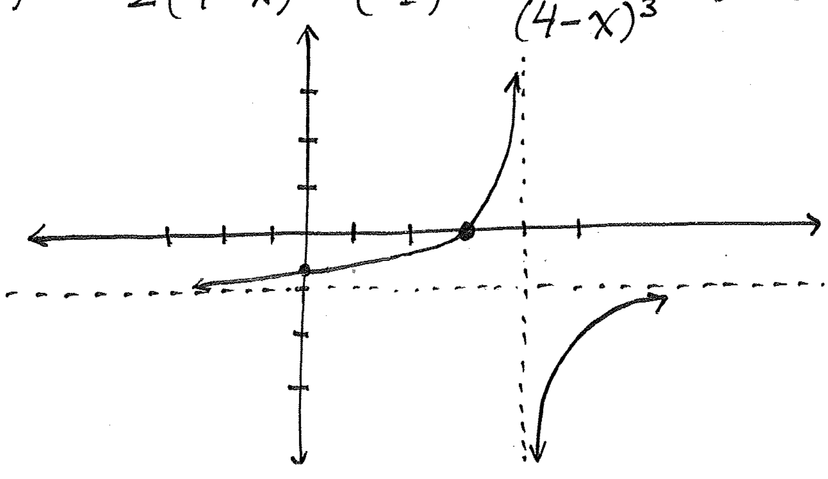
Also note $f(x) = \dfrac{1x + (\text{smaller})}{-1x + (\text{smaller})}$ so when $x \to \pm\infty$, $f(x) \to \dfrac{1}{-1} = -1$.

So $y = -1$ is a horizontal asymptote.

Intercepts: $f(0) = -\frac{3}{4}$, $f(3) = 0$. Derivative: $f' = \dfrac{1(4-x)-(x-3)(-1)}{(4-x)^2}$

$= \dfrac{4-x+x-3}{(4-x)^2} = \dfrac{1}{(4-x)^2}$ never negative. $f' = (4-x)^{-2}$ so then

$f''(x) = -2(4-x)^{-3} \cdot (-1) = \dfrac{2}{(4-x)^3}$ has same sign as $4-x$.



Always increasing
Left part concave up
Right part concave down

2. Draw a graph of the function. Label all the critical points, inflection points, and asymptotes.

$$f(x) = \frac{x}{x^2-4} = \frac{x}{(x+2)(x-2)}$$

Vertical asymptotes at $x=-2,\ x=2$

$f(x) = \dfrac{x + \text{smaller}}{x^2 + \text{smaller}} = \dfrac{\deg 1}{\deg 2}$, so if $x \to \pm\infty$ then $f(x) \to 0$

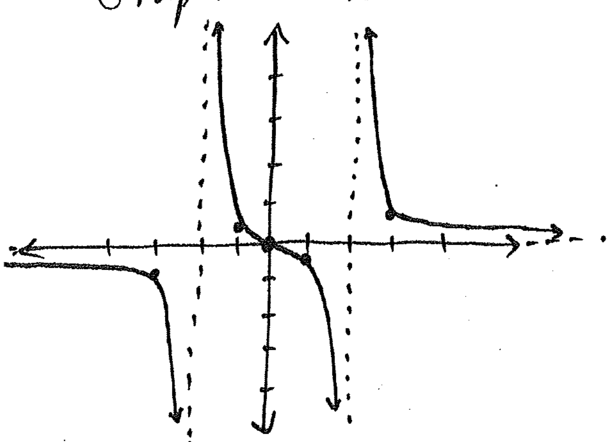
$y=0$ is horizontal asymptote

$f(0)=0$ is only intercept

$$f'(x) = \frac{(x)'(x^2-4)-(x)(x^2-4)'}{(x^2-4)^2} = \frac{1(x^2-4)-x\cdot 2x}{(x^2-4)^2}$$

$$= \frac{x^2-4-2x^2}{(x^2-4)^2} = \frac{-x^2-4}{(x^2-4)^2} = \frac{-(x^2+4)}{(x^2-4)^2}$$ always negative

Graph so far:



$f(2.01) = \dfrac{pos}{pos}$ $\quad f(1.99) = \dfrac{pos}{neg}$

$f(-2.01) = \dfrac{neg}{pos}$ $\quad f(-1.99) = \dfrac{neg}{neg}$

I just guessed when joining points with a curve, but to find inflection points, we need f''. It's hard to fit all that on one page. It would start with

$$f''(x) = -\frac{(x^2+4)'(x^2-4)^2-(x^2+4)((x^2-4)^2)'}{(x^2-4)^4}$$

$$= -\frac{2x(x^2-4)^2-(x^2+4)\cdot 2(x^2-4)\cdot 2x}{(x^2-4)^4}$$

See next page

$$f''(x) = \frac{-2x(x^2-4)\left[(x^2-4) - (x^2+4)\cdot 2\right]}{(x^2-4)^4}$$

$$= \frac{-2x(x^2-4-2x^2-8)}{(x^2-4)^3}$$

$$= \frac{-2x(-x^2-12)}{(x^2-4)^3}$$

$$= \frac{2x(x^2+12)}{(x^2-4)^3}$$

Top has same sign as x
Bottom has same sign as x^2-4

Concavity will change at $x=-2$, $x=0$, $x=2$
(as my graph correctly guessed)

$x=0$ is the only inflection <u>point</u>

3. Find the absolute maximum and absolute minimum of the function

$$f(x) = (x^2 + x)^{2/3}$$

on the interval $[-2, 3]$.

$$f'(x) = \frac{2}{3}(x^2+x)^{-1/3} \cdot (2x+1) = \frac{2(2x+1)}{3(x^2+x)^{1/3}}$$

$f' = 0$ when $x = -\frac{1}{2}$ $\quad f'$ undefined when $x = 0$ or $x = -1$

Check critical points and endpoints.

$f(-2) = (4-2)^{2/3} = 2^{2/3}$ $\qquad f(0) = (0+0)^{2/3} = 0^{2/3} = 0$

$f(-1) = (1-1)^{2/3} = 0^{2/3} = 0$ $\qquad f(3) = (9+3)^{2/3} = 12^{2/3}$

$f(-\frac{1}{2}) = (\frac{1}{4} - \frac{1}{2})^{2/3} = (-\frac{1}{4})^{2/3}$ $\qquad$ Min is 0, max is $12^{2/3}$

4. Find the absolute maximum and absolute minimum of the function

$$f(x) = \frac{x-2}{x+1}$$

on the interval $(-1, 5]$.

$$f'(x) = \frac{(x-2)'(x+1) - (x-2)(x+1)'}{(x+1)^2} = \frac{1(x+1) - (x-2)\cdot 1}{(x+1)^2}$$

$$= \frac{x+1-x+2}{(x+1)^2} = \frac{3}{(x+1)^2}.$$

f' is never 0.
f' is undefined when $x = -1$.

Check critical points and ends of domain.

$$f(5) = \frac{5-2}{5+1} = \frac{3}{6} = \frac{1}{2}$$

$f(\text{near } -1) = \text{e.g. } f(-0.999) = \dfrac{\text{near } -3}{\text{positive near } 0}$ = extreme negative number

Note: x must be slightly more than -1
e.g. $x = -0.999$

Max is $\frac{1}{2}$. There is no min.

5. Find the absolute maximum and absolute minimum of the function

$$f(x) = \frac{\ln x}{x}$$

on the interval $[1, e^2]$. $f'(x) = \dfrac{(\ln x)'x - (\ln x)(x)'}{x^2}$

$= \dfrac{\frac{1}{x}\cdot x - (\ln x)\cdot 1}{x^2} = \dfrac{1 - \ln x}{x^2}$ f' undefined when $x=0$ (not in domain)

$f'=0$ when $\ln x = 1$

$x = e$

Check $f(1), f(e), f(e^2)$.

MORE than $\frac{1}{3}$

LESS than $\frac{1}{3}$

$f(1) = \frac{\ln 1}{1} = \frac{0}{1} = 0$ $f(e) = \frac{\ln e}{e} = \frac{1}{e}$ $f(e^2) = \frac{\ln(e^2)}{e^2} = \frac{2}{e^2}$

Min is 0

Max is $\frac{1}{e} > \frac{1}{3}$

6. Find a number in the closed interval $[\frac{1}{2}, \frac{3}{2}]$ such that the sum of the number and its reciprocal is
a. as small as possible
b. as large as possible.

Let x be the number in $[\frac{1}{2}, \frac{3}{2}]$. Its reciprocal is $\frac{1}{x}$.

Want to minimize or maximize $x + \frac{1}{x} = x + x^{-1}$.

If $f(x) = x + x^{-1}$ then $f'(x) = 1 - x^{-2} = 1 - \frac{1}{x^2}$

$= \frac{x^2-1}{x^2}$. f' undefined when $x=0$ (not in domain)

$f'=0$ when $x = -1$ (not in domain) or $x=1$.

Check $x = \frac{1}{2}, 1, \frac{3}{2}$. $f(\frac{1}{2}) = \frac{1}{2} + 2 = \frac{1}{2} + \frac{4}{2} = \frac{5}{2} = 2.5$

$f(1) = 1+1 = 2$ $f(\frac{3}{2}) = \frac{3}{2} + \frac{2}{3} = \frac{9}{6} + \frac{4}{6} = \frac{13}{6} = 2 + \frac{1}{6}$

Max is 2.5, min is 2. To minimize, choose $x=1$.
To maximize, choose $x = \frac{1}{2}$.

7. How should two nonnegative numbers be chosen so that their sum is 1 and the sum of their squares is

a. as large as possible

b. as small as possible?

We could call the two numbers x and y.

Given: Their sum is 1. $x+y=1$. $y=1-x$

Given: They are nonnegative. $x \geq 0$. $y \geq 0$

$1-x \geq 0$

$1 \geq x \Rightarrow x \leq 1$

Want to minimize or maximize the sum of their squares

i.e. $x^2+y^2 = x^2+(1-x)^2$.

Let $f(x) = x^2+(1-x)^2 = x^2+1-2x+x^2$

$= 2x^2-2x+1$, so $f'(x) = 4x-2$.

Critical points? f' is never undefined. $f'=0$? $4x-2=0$

$4x=2$

$x = \frac{2}{4} = \frac{1}{2}$

Check $x=0$, $x=\frac{1}{2}$, $x=1$.

$f(0) = 0^2+(1-0)^2 = 0+1 = 1$

$f(\frac{1}{2}) = (\frac{1}{2})^2+(1-\frac{1}{2})^2 = \frac{1}{4}+\frac{1}{4} = \frac{1}{2}$

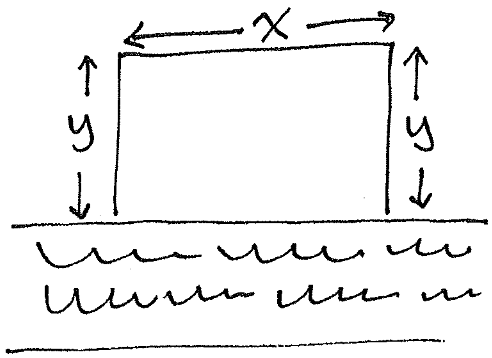
$f(1) = 1^2+(1-1)^2 = 1+0 = 1$

To maximize x^2+y^2, choose $x,y = 0,1$.

To minimize x^2+y^2, choose $x=y=\frac{1}{2}$.

8. A rectangular field is to be bounded by a fence on three sides and by a straight stream on the fourth side. Find the dimensions of the field with maximum area that can be enclosed using 1000 ft of fence.

DRAW A PICTURE and MAKE UP NAMES.



Total length of fence $= x + 2y$

WANT to maximize the area

Area $= xy$

GIVEN: Total length of fence $= 1000$

$$x + 2y = 1000$$
$$2y = 1000 - x$$
$$y = 500 - \tfrac{1}{2}x \Rightarrow \text{Area} = xy = x\left(500 - \tfrac{1}{2}x\right)$$

Area $= 500x - \frac{1}{2}x^2$. Call this $f(x)$. We want to maximize $f(x)$.

$f'(x) = 500 - x$. Critical numbers? f' is never undefined. $f' = 0$ if $x = 500$.

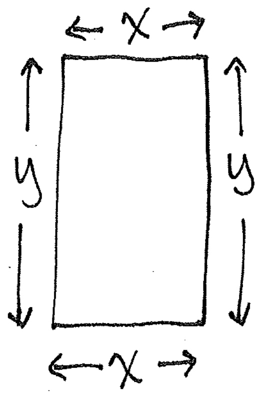
Check the critical number $x = 500$ and the ends of the domain which are the extreme cases: $x = 0$ or $x = 1000$.

$$f(0) = 0 \cdot \left(500 - \tfrac{1}{2} \cdot 0\right) = 0$$
$$f(500) = 500 \cdot \left(500 - \tfrac{1}{2} \cdot 500\right) = 500 \cdot 250 = 125{,}000$$
$$f(1000) = 1000 \cdot \left(500 - \tfrac{1}{2} \cdot 1000\right) = 0$$

The field should be 500 ft by 250 ft

9. A rectangular plot of land is to be fenced in using two kinds of fencing. Two opposite sides will use heavy-duty fencing selling for \$3 a foot, while the remaining two sides will use standard fencing selling for \$2 a foot. What are the dimensions of the rectangular plot of greatest area that can be fenced in for a cost of \$6000?

DRAW A PICTURE and MAKE UP NAMES



Suppose the sides of length x are heavy-duty fencing (\$3 per foot)
Suppose the sides of length y are standard fencing (\$2 per foot)

Want to MAXIMIZE the area. Area $= xy$

Total cost $=$ \$6000.

Total length of heavy-duty fence $= 2x$ feet.
Total cost of heavy-duty fence $= (2x \text{ feet}) \cdot (3 \text{ dollars per foot})$
$= 6x$ dollars.

Similarly, total cost of standard fence $= 2y \cdot 2 = 4y$.

So, $6x + 4y = 6000 \Rightarrow 3x + 2y = 3000$

$$2y = 3000 - 3x$$
$$y = 1500 - \frac{3}{2}x$$

$$\text{Area} = xy = x\left(1500 - \frac{3}{2}x\right) = 1500x - \frac{3}{2}x^2$$

Call this $f(x)$.
Want to maximize $f(x)$.
NEXT PAGE

$$f(x) = 1500x - \frac{3}{2}x^2$$

$$f'(x) = 1500 - 3x$$

Critical numbers? f' is never undefined

$f' = 0$ when $x = 500$

Check $x = 500$ and the ends of the domain.

WHAT ARE the ends of the domain?

What are restrictions on x or y? Total cost $= 6000$

Cost of heavy-duty fence is $6x$, so $6x \leq 6000$

$x \leq 1000$

Also $x \geq 0$ because x is a length.

Check $x = 0$, $x = 500$, $x = 1000$.

$$f(0) = 0 \cdot \left(1500 - \frac{3}{2} \cdot 0\right) = 0$$

$$f(500) = 500 \cdot \left(1500 - \frac{3}{2} \cdot 500\right)$$
$$= 500 \cdot (1500 - 750) = 500 \cdot 750$$

$$f(1000) = 1000 \cdot \left(1500 - \frac{3}{2} \cdot 1000\right)$$
$$= 1000 \cdot (1500 - 1500) = 0$$

Area is maximized when $x = 500$ and $y = 750$.

10. Evaluate the integral.

$$\int (x^{-3} - 3x^{1/4} + 8x^2)\,dx$$

$$= \int x^{-3}\,dx - \int 3x^{1/4}\,dx + \int 8x^2\,dx$$

$$= \int x^{-3}\,dx - 3\int x^{1/4}\,dx + 8\int x^2\,dx$$

$$= \frac{x^{-3+1}}{-3+1} - 3\cdot\frac{x^{\frac{1}{4}+1}}{\frac{1}{4}+1} + 8\cdot\frac{x^3}{3} + C$$

$$\boxed{\begin{gathered}\frac{x^{-2}}{-2} - 3\cdot\frac{x^{5/4}}{5/4} + \frac{8x^3}{3} + C \\ \text{or} \\ -\frac{1}{2x^2} - 3\cdot\frac{4}{5}x^{5/4} + \frac{8x^3}{3} + C\end{gathered}}$$

11. Evaluate the integral.

$$\int x(1+x^3)\,dx = \int (x + x^4)\,dx$$

$$= \frac{x^2}{2} + \frac{x^5}{5} + C$$

12. Evaluate the integral.

$$\int \frac{x^5 + 2x^2 - 1}{x^4}\, dx$$

$$\int \left(\frac{x^5}{x^4} + \frac{2x^2}{x^4} - \frac{1}{x^4} \right) dx = \int \left(x + 2x^{-2} - x^{-4} \right) dx$$

$$= \frac{x^2}{2} + 2 \cdot \frac{x^{-1}}{-1} - \frac{x^{-3}}{-3} + C$$

or $\frac{x^2}{2} - \frac{2}{x} + \frac{1}{x^3} + C$

13. Evaluate the integral.

$$\int (3\sin x - 2\sec^2 x)\, dx$$

$$= \int 3 \sin x \, dx - \int 2 \sec^2 x \, dx$$

$$= 3 \int \sin x \, dx - 2 \int \sec^2 x \, dx$$

$$= 3(-\cos x) - 2 \tan x + C$$

or $-3\cos x - 2\tan x + C$

14. Evaluate the integral.

$$\int \sec x(\sec x + \tan x)\, dx$$

$$= \int \left(\sec^2 x + \sec x \tan x\right) dx$$

$$= \int \sec^2 x\, dx + \int \sec x \tan x\, dx$$

$$= \tan x + \sec x + C$$

15. Evaluate the integral.

$$\int 2x(x^2+1)^{23}\, dx$$

Try substitution. Try $u = x^2 + 1$

$$\Rightarrow \frac{du}{dx} = 2x \Rightarrow du = 2x\, dx$$

$$\text{Integral} = \int \underbrace{(x^2+1)^{23}}_{u^{23}} \cdot \underbrace{2x\, dx}_{du} = \int u^{23}\, du$$

$$= \frac{u^{24}}{24} + C = \frac{(x^2+1)^{24}}{24} + C$$

16. Evaluate the integral.

$$\int \cos^3 x \sin x \, dx$$

Try substitution. Try $u = \cos x \Rightarrow \frac{du}{dx} = -\sin x$

$\Rightarrow du = -\sin x \, dx$

$\Rightarrow -du = \sin x \, dx$

$$\text{Integral} = \int \underbrace{(\cos x)^3}_{u^3} \underbrace{\sin x \, dx}_{-du}$$

$$= \int -u^3 \, du = -\frac{u^4}{4} + C = -\frac{\cos^4 x}{4} + C$$

17. Evaluate the integral.

$$\int \sec^2(4x+1) \, dx$$

Try substitution. Try $u = 4x+1 \Rightarrow \frac{du}{dx} = 4$

$\Rightarrow du = 4dx \Rightarrow \frac{1}{4} du = dx$

$$\text{Integral} = \int \underbrace{\sec^2(4x+1)}_{\sec^2 u} \underbrace{dx}_{\frac{1}{4}du} = \int \frac{1}{4} \sec^2 u \, du$$

$$= \frac{1}{4} \int \sec^2 u \, du = \frac{1}{4} \tan u + C$$

$$= \frac{1}{4} \tan(4x+1) + C$$

18. Evaluate the integral.

$$\int \frac{3x}{\sqrt{4x^2+5}}\,dx$$

Try $u = 4x^2 + 5 \Rightarrow \frac{du}{dx} = 8x \Rightarrow du = 8x\,dx$

$\Rightarrow \frac{1}{8}du = x\,dx$. Integral $= \int 3 \cdot \underbrace{(4x^2+5)^{-1/2}}_{u^{-1/2}} \cdot \underbrace{x\,dx}_{\frac{1}{8}du}$

$= \int \frac{3}{8} u^{-1/2}\,du = \frac{3}{8} \cdot \frac{u^{1/2}}{1/2} + C = \frac{3}{8} \cdot \frac{2}{1} \cdot u^{1/2} + C$

$= \frac{3}{4}\sqrt{u} + C$

$= \frac{3}{4}\sqrt{4x^2+5} + C.$

19. Evaluate the integral.

$$\int (4x-3)^9\,dx$$

Sub $u = 4x - 3$

$\frac{du}{dx} = 4 \Rightarrow du = 4\,dx$

$\frac{1}{4}du = dx$

$\int (4x-3)^9\,dx = \int u^9 \cdot \frac{1}{4}\,du = \frac{1}{4} \cdot \frac{u^{10}}{10} + C$

$= \frac{(4x-3)^{10}}{40} + C.$

20. Evaluate the integral.

$$\int \sec 4x \tan 4x \, dx$$

Try $u = 4x \Rightarrow \frac{du}{dx} = 4 \Rightarrow du = 4dx$

$\frac{1}{4} du = dx.$

$$\int \sec 4x \tan 4x \, dx = \int \sec u \tan u \, \frac{1}{4} du$$

$$= \frac{1}{4} \sec u + C = \frac{1}{4} \sec 4x + C$$

21. Evaluate the integral.

$$\int e^{2x} \, dx$$

Try $u = 2x \Rightarrow \frac{du}{dx} = 2 \Rightarrow du = 2dx$

$\frac{1}{2} du = dx$

$$\int e^{2x} dx = \int e^{u} \cdot \frac{1}{2} du = \frac{1}{2} \int e^{u} du$$

$$= \frac{1}{2} e^{u} + C = \frac{1}{2} e^{2x} + C$$

22. Evaluate the integral.

$$\int x^2 e^{-2x^3}\,dx$$

Try the substitution $u = -2x^3$

Then $\frac{du}{dx} = -6x^2$

$du = -6x^2\,dx$

$-\frac{1}{6}\,du = x^2\,dx$

$$\int e^{-2x^3} x^2\,dx = \int e^u \cdot \left(-\frac{1}{6}\right) du$$

$$= -\frac{1}{6}\int e^u\,du = -\frac{1}{6}e^u + C$$

$$= -\frac{1}{6}e^{-2x^3} + C$$