Recall: $f(x) \approx f(x_0) + f'(x_0)\cdot(x-x_0)$

$$\underbrace{f(x)-f(x_0)}_{\text{change in } f} \approx \underbrace{f'(x_0)}_{\substack{\text{Derivative} \\ \text{(at "nice" value)}}}\cdot\underbrace{(x-x_0)}_{\text{change in } x}$$

WRITE YOUR NAME:

MAC 2311 Homework 6
Due in class, Friday March 31st
You can use more paper if necessary, but please STAPLE

Question 1. Find the local linear approximation of the function $f(x) = \sqrt{x}$ at $x_0 = 9$, and use it to estimate $\sqrt{8}$ and $\sqrt{10}$.

$$f(x) = \sqrt{x} = x^{1/2}$$

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$f(x_0) = f(9) = 9^{1/2} = 3$$

$$f'(x_0) = f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{2\cdot 3} = \frac{1}{6}$$

General formula for linear approximation is

$$f(x) \approx f(x_0) + f'(x_0)\cdot(x-x_0)$$
$$= 3 + \frac{1}{6}(x-9)$$

So $\sqrt{8} = f(8) \approx 3 + \frac{1}{6}(8-9) = 3 - \frac{1}{6}$

$\sqrt{10} = f(10) \approx 3 + \frac{1}{6}(10-9) = 3 + \frac{1}{6}$

$\frac{1}{6} \approx 0.1667$ so $\sqrt{8} \approx 2.8333$ and $\sqrt{10} \approx 3.1667$

Question 2. Find the local linear approximation of the function $f(x) = \sqrt{1+x}$ at $x_0 = 0$, and use it to estimate $\sqrt{0.9}$ and $\sqrt{1.1}$.

$$f(x) = (1+x)^{1/2}$$

$$f'(x) = \frac{1}{2}(1+x)^{-1/2} \cdot 1 = \frac{1}{2(1+x)^{1/2}}$$

$$f(x_0) = f(0) = (1+0)^{1/2} = 1^{1/2} = 1$$

$$f'(x_0) = f'(0) = \frac{1}{2(1+0)^{1/2}} = \frac{1}{2 \cdot 1} = \frac{1}{2}$$

$$f(x) \approx f(x_0) + f'(x_0) \cdot (x - x_0)$$

$$= 1 + \frac{1}{2}(x-0) = 1 + \frac{x}{2}$$

So

$$\sqrt{0.9} = \sqrt{1+(-0.1)} = f(-0.1) \approx 1 + \frac{-0.1}{2}$$

$$= 1 + (-0.05) = 0.95$$

$$\sqrt{1.1} = \sqrt{1+(0.1)} = f(0.1) \approx 1 + \frac{0.1}{2}$$

$$= 1 + 0.05 = 1.05$$

Question 3. Evaluate the limit.

$$\lim_{x \to 0} \frac{x^2}{\sin x}$$

If x approaches 0, then the top x^2 approaches 0
and the bottom $\sin x$ approaches 0

So this limit problem has the "form" $\frac{0}{0}$

and we can use L'Hopital's rule

$$\lim_{x \to 0} \frac{x^2}{\sin x} = \lim_{x \to 0} \frac{2x}{\cos x}$$

$$= \frac{2 \cdot 0}{\cos 0} = \frac{0}{1} = 0.$$

Question 4. Evaluate the limit.

$$\lim_{x\to 0} \frac{e^x - 1}{\sin x}$$

Note $e^0 - 1 = 1 - 1 = 0$

$\sin 0 = 0$

Limit has the form $\frac{0}{0}$ $\Rightarrow$ L'Hopital

$$\lim_{x\to 0} \frac{e^x - 1}{\sin x} = \lim_{x\to 0} \frac{e^x}{\cos x}$$

$$= \frac{e^0}{\cos 0} = \frac{1}{1} = 1$$

Question 5. Evaluate the limit.

$$\lim_{x \to 0} \frac{\sin 2x}{\sin 5x}$$

Note $\sin(2 \cdot 0) = \sin 0 = 0$

$\sin(5 \cdot 0) = \sin 0 = 0$

Limit is of form $\frac{0}{0}$ $\Rightarrow$ L'Hopital

$$\lim_{x \to 0} \frac{\sin 2x}{\sin 5x} = \lim_{x \to 0} \frac{\cos(2x) \cdot 2}{\cos(5x) \cdot 5}$$

$$= \frac{\cos(2 \cdot 0) \cdot 2}{\cos(5 \cdot 0) \cdot 5} = \frac{\cos(0) \cdot 2}{\cos(0) \cdot 5}$$

$$= \frac{1 \cdot 2}{1 \cdot 5} = \frac{2}{5}$$

Question 6. Evaluate the limit.

$$\lim_{x\to 0}(1-3x)^{1/x}$$

Let $L = (1-3x)^{1/x}$. We want $\lim\limits_{x\to 0} L$.

$$\ln L = \ln\left((1-3x)^{1/x}\right) = \frac{1}{x}\ln(1-3x) = \frac{\ln(1-3x)}{x}$$

Note $\ln(1-0) = \ln 1 = 0$ so $\ln L = \dfrac{\ln(1-3x)}{x}$ has $\frac{0}{0}$ form

so we can use L'Hopital.

$$\lim_{x\to 0}\ln L = \lim_{x\to 0}\frac{\ln(1-3x)}{x} = \lim_{x\to 0}\frac{\frac{1}{1-3x}\cdot(-3)}{1}$$

$$= \lim_{x\to 0}\frac{-3}{1-3x} = \frac{-3}{1-0} = -3.$$

So $\ln L$ approaches -3

so L approaches e^{-3}.

Answer: $\lim\limits_{x\to 0} L = \lim\limits_{x\to 0}(1-3x)^{1/x} = e^{-3}$.