MAC2311 Section U08
Suggested problems for Test 1
(Test 1 is Friday February 10th, in class)

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1. Evaluate the limit.

$$\lim_{x\to 0} \frac{6x-9}{x^3-12x+3}$$

Nothing prevents us from plugging in $x=0$

$$\frac{6\cdot 0-9}{0^3-12\cdot 0+3} = \frac{0-9}{0-0+3} = \frac{-9}{3} = -3$$

2. Evaluate the limit.

$$\lim_{x\to 1} \frac{x^4-1}{x-1}$$

Factor a difference of squares (twice)

$$\lim_{x\to 1} \frac{(x^2-1)(x^2+1)}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)(x^2+1)}{x-1}$$

$$= \lim_{x\to 1} (x+1)(x^2+1)$$

$$= (1+1)(1+1)$$

$$= 2\cdot 2 = 4$$

3. Evaluate the limit.

$$\lim_{x\to 2} \frac{x^2-4x+4}{x^2+x-6}$$

$$\lim_{x\to 2} \frac{(x-2)(x-2)}{(x-2)(x+3)} = \lim_{x\to 2} \frac{x-2}{x+3}$$

$$= \frac{2-2}{2+3} = \frac{0}{5} = 0$$

4. Evaluate the limit.

$$\lim_{x\to 3^+} \frac{x}{x-3}$$

$\frac{\text{NONZERO}}{\text{ZERO}}$

$x \to 3^+$
$x > 3$
$x - 3 > 0$

Top approaches 3, which is positive
Bottom approaches 0 and is positive

Answer: $+\infty$ or "does not exist"

5. Evaluate the limit.

$$\lim_{x\to 3^-} \frac{x}{x-3}$$

$\frac{\text{NONZERO}}{\text{ZERO}}$

$x \to 3^-$
$x < 3$
$x - 3 < 0$

Top approaches 3, which is positive
Bottom approaches 0 and is negative

Answer: $-\infty$ or "does not exist"

6. Evaluate the limit.

$$\lim_{x\to 2^+} \frac{x}{x^2-4} = \lim_{x\to 2^+} \frac{x}{(x-2)(x+2)}$$

$\frac{\text{NONZERO}}{\text{ZERO}}$

When $x \to 2^+$, x is positive and approaches 2
$x-2$ is positive and approaches 0
$x+2$ is positive and approaches 4

Answer: $+\infty$ or "does not exist"

7. Evaluate the limit.

$$\lim_{x\to 2^-} \frac{x}{x^2-4} = \lim_{x\to 2^-} \frac{x}{(x-2)(x+2)}$$

$\frac{\text{NONZERO}}{\text{ZERO}}$

When $x \to 2^-$, x is positive and approaches 2
$x-2$ is negative and approaches 0
$x+2$ is positive and approaches 4

So whole thing is $\frac{\text{POS}}{\text{NEG} \cdot \text{POS}} = \text{NEG}$ and also $\frac{\text{NONZERO}}{\text{ZERO}}$

Answer: $-\infty$ or "does not exist"

$\frac{\text{zero}}{\text{zero}} \to$ don't know yet...

8. Evaluate the limit.

$$\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3}$$

Try multiplying by "conjugate"

$$\lim_{x\to 9} \frac{(x-9)(\sqrt{x}+3)}{(\sqrt{x}-3)(\sqrt{x}+3)} = \lim_{x\to 9} \frac{(x-9)(\sqrt{x}+3)}{(\sqrt{x})^2-3^2}$$

$$= \lim_{x\to 9} \frac{(x-9)(\sqrt{x}+3)}{x-9} = \lim_{x\to 9} (\sqrt{x}+3) = \sqrt{9}+3$$

$$= 3+3 = 6$$

9. Evaluate the limit.

Conjugate?

$$\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x}$$

$\frac{\text{zero}}{\text{zero}}$ Don't know yet...

$$\lim_{x\to 0} \frac{(\sqrt{x+4}-2)(\sqrt{x+4}+2)}{x(\sqrt{x+4}+2)} = \lim_{x\to 0} \frac{(x+4)-4}{x(\sqrt{x+4}+2)}$$

$$= \lim_{x\to 0} \frac{x}{x(\sqrt{x+4}+2)} = \lim_{x\to 0} \frac{1}{\sqrt{x+4}+2} = \frac{1}{\sqrt{0+4}+2}$$

$$= \frac{1}{\sqrt{4}+2} = \frac{1}{2+2} = \frac{1}{4}$$

10. Evaluate the limit.

$$\lim_{x\to\infty} \frac{3x+1}{2x-5}$$

$$\lim_{x\to\infty} \frac{(3x+1)\cdot\frac{1}{x}}{(2x-5)\cdot\frac{1}{x}} = \lim_{x\to\infty} \frac{3+\frac{1}{x}}{2-\frac{5}{x}}$$

$$= \frac{3+0}{2-0} = \frac{3}{2}$$

11. Evaluate the limit.

$$\lim_{y \to -\infty} \frac{3}{y+4}$$

$\dfrac{\text{Finite}}{\text{Infinite}}$

0

12. Evaluate the limit.

$$\lim_{x \to -\infty} \frac{x-2}{x^2+2x+1}$$

$$\lim_{x \to -\infty} \frac{(x-2)\cdot \frac{1}{x^2}}{(x^2+2x+1)\cdot \frac{1}{x^2}} = \lim_{x \to -\infty} \frac{\frac{1}{x} - \frac{2}{x^2}}{1 + \frac{2}{x} + \frac{1}{x^2}}$$

$$= \frac{0-0}{1+0+0} = \frac{0}{1} = 0$$

13. Evaluate the limit.

$$\lim_{x \to \infty} \frac{5x^2+7}{3x^2-x}$$

$$\lim_{x \to \infty} \frac{(5x^2+7)\cdot \frac{1}{x^2}}{(3x^2-x)\cdot \frac{1}{x^2}} = \lim_{x \to \infty} \frac{5 + \frac{7}{x^2}}{3 - \frac{1}{x}} = \frac{5+0}{3-0} = \frac{5}{3}$$

14. Evaluate the limit.

$$\lim_{t \to -\infty} \frac{5-2t^3}{t^2+1}$$

$$\lim_{t \to -\infty} \frac{(5-2t^3)\cdot \frac{1}{t^3}}{(t^2+1)\cdot \frac{1}{t^3}} = \lim_{t \to -\infty} \frac{\frac{5}{t^3} - 2}{\frac{1}{t} + \frac{1}{t^3}}$$

Approaches $\dfrac{0-2}{0+0} = \dfrac{-2}{0}$ $\dfrac{\text{Nonzero}}{\text{Zero}}$ Limit does not exist.

Also, if $t \to -\infty$, then $\dfrac{5-2t^3}{t^2+1} \approx \dfrac{-2t^3}{t^2} = -2t \to +\infty$

15. Evaluate the limit.

$$\lim_{x\to-\infty}\frac{x+4x^3}{1-x^2+7x^3}$$

$$\lim_{x\to-\infty}\frac{(x+4x^3)\cdot\frac{1}{x^3}}{(1-x^2+7x^3)\cdot\frac{1}{x^3}} = \lim_{x\to-\infty}\frac{\frac{1}{x^2}+4}{\frac{1}{x^3}-\frac{1}{x}+7}$$

$$= \frac{0+4}{0-0+7} = \frac{4}{7}$$

16. Evaluate the limit.

$$\lim_{x\to\infty}\sqrt{\frac{2-3x+4x^2}{1+9x^2}}$$

$$\lim_{x\to\infty}\sqrt{\frac{(2-3x+4x^2)\cdot\frac{1}{x^2}}{(1+9x^2)\cdot\frac{1}{x^2}}} = \lim_{x\to\infty}\sqrt{\frac{\frac{2}{x^2}-\frac{3}{x}+4}{\frac{1}{x^2}+9}}$$

$$= \sqrt{\frac{0-0+4}{0+9}} = \sqrt{\frac{4}{9}} = \frac{2}{3}$$

17. Evaluate the limit.

$$\lim_{x\to\infty}\frac{\sqrt{5x^2-2}}{x+3}$$

Idea: Top "really" grows like $\sqrt{x^2}=x$

$$\lim_{x\to\infty}\frac{\sqrt{5x^2-2}\cdot\frac{1}{x}}{(x+3)\cdot\frac{1}{x}} = \lim_{x\to\infty}\frac{\sqrt{5x^2-2}\cdot\sqrt{\frac{1}{x^2}}}{(x+3)\cdot\frac{1}{x}}$$

$$= \lim_{x\to\infty}\frac{\sqrt{(5x^2-2)\cdot\frac{1}{x^2}}}{(x+3)\cdot\frac{1}{x}} = \lim_{x\to\infty}\frac{\sqrt{5-\frac{2}{x^2}}}{1+\frac{3}{x}}$$

$$= \frac{\sqrt{5-0}}{1+0} = \frac{\sqrt{5}}{1} = \sqrt{5}$$

18. Find all values of x, if any, at which f is not continuous.

$$f(x) = (x-8)^{1/3}$$

Continuous for all x.

(Composition of $x-8$, which is continuous everywhere, and cube root function, which is continuous everywhere.)

19. Find all values of x, if any, at which f is not continuous.

$$f(x) = \frac{x+2}{x^2-4}$$

Discontinuous when $x^2-4=0$ $\quad x^2=4 \quad x=\pm 2$

(Alternatively: $x^2-4=0 \Rightarrow (x-2)(x+2)=0 \Rightarrow x=2$ or -2)

20. Find all values of x, if any, at which f is not continuous.

$$f(x) = \frac{x}{2x^2+x}$$

Discontinuous when $2x^2+x=0$

$x(2x+1)=0$

$x=0$ or $2x+1=0 \Rightarrow 2x=-1 \Rightarrow x=\frac{-1}{2}$

21. Find all values of x, if any, at which f is not continuous.

$$f(x) = \frac{3}{x} + \frac{x-1}{x^2-1}$$

Discontinuous when $x=0$ or $x^2-1=0$

$x^2=1$

$x=1$ or -1

Discontinuous if $x=0$ or 1 or -1

22. Find all values of x (if any) at which f is not continuous, and determine whether each discontinuity is a removable discontinuity.

$$f(x) = \frac{x^2+3x}{x+3}$$ Discontinuous at $x = -3$.

$$\lim_{x \to -3} \frac{x^2+3x}{x+3} = \lim_{x \to -3} \frac{x(x+3)}{x+3} = \lim_{x \to -3} x = -3$$

Limit exists
Removable.

23. Find the discontinuities, if any.

$$f(x) = \sin(x^2 - 2)$$

No discontinuities. Composition of polynomial x^2-2, which is continuous everywhere, and sine function, which is continuous everywhere.

24. Find the discontinuities, if any.

$$f(x) = \cos\left(\frac{x}{x-\pi}\right)$$

Cosine function is continuous everywhere, but $\frac{x}{x-\pi}$ is discontinuous when $x = \pi$.

25. Find the discontinuities, if any.

$$f(x) = |\cot x|$$ Recall $\cot x = \frac{\cos x}{\sin x}$

Discontinuous when $\sin x = 0$, i.e. when $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

26. Find the discontinuities, if any.

$$f(x) = \frac{1}{1+\sin^2 x}$$

No discontinuities. Discontinuous if $1 + \sin^2 x = 0$

i.e. if $\sin^2 x = -1$

$\sin x = \sqrt{-1}$ Undefined if using real numbers

Never happens

27. Find the limit.

$$\lim_{x\to\infty} \cos\left(\frac{1}{x}\right)$$

Since $\lim_{x\to\infty} \frac{1}{x} = 0$, we conclude $\lim_{x\to\infty} \cos\left(\frac{1}{x}\right)$

$= \cos 0 = 1$

28. Find the limit.

$$\lim_{x\to\infty} \sin\left(\frac{\pi x}{2-3x}\right)$$

$$\lim_{x\to\infty} \sin\left(\frac{\pi x \cdot \frac{1}{x}}{(2-3x)\cdot\frac{1}{x}}\right) = \lim_{x\to\infty} \sin\left(\frac{\pi}{\frac{2}{x}-3}\right)$$

$$= \sin\left(\frac{\pi}{0-3}\right) = \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

29. Find the limit.

$$\lim_{x\to\infty} \sin^{-1}\left(\frac{x}{1-2x}\right)$$

$$\lim_{x\to\infty} \sin^{-1}\left(\frac{x\cdot\frac{1}{x}}{(1-2x)\cdot\frac{1}{x}}\right) = \lim_{x\to\infty} \sin^{-1}\left(\frac{1}{\frac{1}{x}-2}\right)$$

$$= \sin^{-1}\left(\frac{1}{0-2}\right) = \sin^{-1}\left(-\frac{1}{2}\right)$$

$$= -\pi/6$$

30. Find the limit.

$$\lim_{x\to\infty} \ln\left(\frac{x+1}{x}\right)$$

$$\lim_{x\to\infty} \ln\left(\frac{(x+1)\cdot\frac{1}{x}}{x\cdot\frac{1}{x}}\right) = \lim_{x\to\infty} \ln\left(\frac{1+\frac{1}{x}}{1}\right)$$

$$= \ln\left(\frac{1+0}{1}\right) = \ln\left(\frac{1}{1}\right) = \ln 1 = 0$$

31. Write down the **definition** of the derivative.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

32. Use the **definition** of the derivative to find the derivative of $f(x) = x^2$.

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h}$$

$$= \lim_{h\to 0} \frac{2xh+h^2}{h} = \lim_{h\to 0} \frac{(2x+h)h}{h} = \lim_{h\to 0} (2x+h)$$

$$= 2x+0 = 2x$$

33. Use the **definition** of the derivative to find the derivative of $f(x) = \frac{1}{x}$.

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \cdot \frac{(x+h)x}{(x+h)x}$$

$$= \lim_{h\to 0} \frac{x-(x+h)}{h(x+h)x} = \lim_{h\to 0} \frac{-h}{h(x+h)x}$$

$$= \lim_{h\to 0} \frac{-1}{(x+h)x} = \frac{-1}{(x+0)x} = \frac{-1}{x^2}$$