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MAC 2311 Section U08 Test 2
Friday March 3rd
Total possible score: 20 points (2 points per page)

Question 1. Find dy/dx.

$$y = 4x^7 + 2x^3 - 89$$

$$\frac{dy}{dx} = 4 \cdot 7x^6 + 2 \cdot 3x^2 + 0$$

$$= 28x^6 + 6x^2$$

Question 2a. Write down the **definition** of the derivative.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Question 2b. Use the **definition** of the derivative to find the derivative of $f(x) = x^3$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)(x+h)(x+h) - x^3}{h} = \lim_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h \to 0} \frac{(3x^2 + 3xh + h^2)h}{h}$$

$$= \lim_{h \to 0} (3x^2 + 3xh + h^2)$$

$$= 3x^2 + 0 + 0 = 3x^2$$

Question 3. Find $f'(x)$.

$$f(x) = \frac{x + 2x^{3/2}}{\sqrt{x}}$$

Easiest method:

$$f(x) = \frac{x + 2x^{3/2}}{x^{1/2}} = \frac{x}{x^{1/2}} + \frac{2x^{3/2}}{x^{1/2}}$$

$$= x^{1/2} + 2x, \text{ so } f'(x) = \frac{1}{2}x^{-1/2} + 2$$

$$\text{or } \frac{1}{2\sqrt{x}} + 2 \quad \text{or } \frac{1 + 4\sqrt{x}}{2\sqrt{x}}$$

Harder method (but still correct): Use quotient rule

$$f'(x) = \frac{(x + 2x^{3/2})'(\sqrt{x}) - (x + 2x^{3/2})(\sqrt{x})'}{(\sqrt{x})^2}$$

$$= \frac{(1 + 2 \cdot \frac{3}{2}x^{1/2})\sqrt{x} - (x + 2x^{3/2}) \cdot \frac{1}{2}x^{-1/2}}{x}$$

Technically can stop there. Can also simplify.

$$\frac{(1 + 3\sqrt{x})\sqrt{x} - (x + 2x^{3/2}) \cdot \frac{1}{2}x^{-1/2}}{x} = \frac{\sqrt{x} + 3x - (\frac{1}{2}\sqrt{x} + x)}{x}$$

$$= \frac{\frac{1}{2}\sqrt{x} + 2x}{x} = \frac{\sqrt{x} + 4x}{2x}$$

Question 4. Find the **second** derivative $f''(x)$.

$$f(x) = \frac{3x+7}{2x+5}$$

$$f'(x) = \frac{(3x+7)'(2x+5) - (3x+7)(2x+5)'}{(2x+5)^2}$$

$$= \frac{3(2x+5) - (3x+7)2}{(2x+5)^2} = \frac{6x+15-(6x+14)}{(2x+5)^2}$$

$$= \frac{6x+15-6x-14}{(2x+5)^2} = \frac{1}{(2x+5)^2} \quad \text{or } (2x+5)^{-2}$$

$$f''(x) = -2(2x+5)^{-3} \cdot (2x+5)' \quad \text{CHAIN RULE}$$

$$= -2(2x+5)^{-3} \cdot 2 = -4(2x+5)^{-3}$$

$$\text{or } \frac{-4}{(2x+5)^3}$$

COULD also use quotient rule to find derivative of $\frac{1}{(2x+5)^2}$

$$\frac{1'(2x+5)^2 - 1((2x+5)^2)'}{((2x+5)^2)^2} = \frac{0 - 2(2x+5)\cdot 2}{(2x+5)^4} = \frac{-4}{(2x+5)^3}$$

Question 5. Find $f'(x)$ and **simplify**.

$$f(x) = \frac{5 - \cos x}{5 + \sin x}$$

$$f'(x) = \frac{(5-\cos x)'(5+\sin x) - (5-\cos x)(5+\sin x)'}{(5+\sin x)^2}$$

$$= \frac{\sin x \cdot (5+\sin x) - (5-\cos x)\cos x}{(5+\sin x)^2}$$

$$= \frac{5\sin x + \sin^2 x - (5\cos x - \cos^2 x)}{(5+\sin x)^2}$$

$$= \frac{5\sin x + \sin^2 x - 5\cos x + \cos^2 x}{(5+\sin x)^2}$$

$$= \frac{5\sin x - 5\cos x + 1}{(5+\sin x)^2}$$

Question 6. Find $f'(x)$ using any correct method.

$$f(x) = \left(\frac{x^2-1}{x^2+1}\right)^{17}$$

Easiest method is probably to take logarithms first

$$\ln f(x) = \ln\left(\left(\frac{x^2-1}{x^2+1}\right)^{17}\right) = 17\ln\left(\frac{x^2-1}{x^2+1}\right)$$

$$\ln f(x) = 17\left(\ln(x^2-1) - \ln(x^2+1)\right)$$

$$\frac{d}{dx}\left(\ln f(x)\right) = \frac{d}{dx}\left(17\left(\ln(x^2-1) - \ln(x^2+1)\right)\right)$$

$$\frac{1}{f(x)} \cdot f'(x) = 17\left(\frac{1}{x^2-1}\cdot 2x - \frac{1}{x^2+1}\cdot 2x\right)$$

$$f'(x) = 17f(x)\cdot\left(\frac{2x}{x^2-1} - \frac{2x}{x^2+1}\right)$$

$$\text{or } 17\left(\frac{x^2-1}{x^2+1}\right)^{17}\left(\frac{2x}{x^2-1} - \frac{2x}{x^2+1}\right)$$

Alternate method: $f'(x) = 17\left(\frac{x^2-1}{x^2+1}\right)^{16}\cdot\left(\frac{x^2-1}{x^2+1}\right)'$

$$= 17\left(\frac{x^2-1}{x^2+1}\right)^{16}\cdot\frac{(x^2-1)'(x^2+1) - (x^2-1)(x^2+1)'}{(x^2+1)^2}$$

$$= 17\left(\frac{x^2-1}{x^2+1}\right)^{16}\cdot\frac{2x(x^2+1) - (x^2-1)\cdot 2x}{(x^2+1)^2} = 17\left(\frac{x^2-1}{x^2+1}\right)^{16}\cdot\frac{4x}{(x^2+1)^2}$$

Question 7. Find $f'(x)$.

$$f(x) = \cos^3\left(\frac{x}{x+1}\right)$$

or equivalently $$f(x) = \left(\cos\left(\frac{x}{x+1}\right)\right)^3$$

$f = u^3$ and $u = \cos\left(\frac{x}{x+1}\right)$

$u = \cos v$ and $v = \frac{x}{x+1}$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$\frac{df}{du} = 3u^2$

$\frac{du}{dv} = -\sin v$

$$\frac{dv}{dx} = \frac{(x)'(x+1) - x(x+1)'}{(x+1)^2} = \frac{1(x+1) - x \cdot 1}{(x+1)^2} = \frac{1}{(x+1)^2}$$

$$f'(x) = \frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

$$= 3u^2 \cdot (-\sin v) \cdot \frac{1}{(x+1)^2}$$

$$= 3\cos^2\left(\frac{x}{x+1}\right) \cdot \left(-\sin\frac{x}{x+1}\right) \cdot \frac{1}{(x+1)^2}$$

Question 8. Find dy/dx using any correct method.

$$x^4 + y^4 = 4xy^3$$

$$\frac{d}{dx}\left(x^4 + y^4\right) = \frac{d}{dx}\left(4xy^3\right)$$

$$4x^3 + 4y^3\frac{dy}{dx} = (4x)'y^3 + 4x(y^3)'$$

$$4x^3 + 4y^3y' = 4y^3 + 4x\cdot 3y^2\cdot y'$$

$$4y^3y' - 12xy^2y' = 4y^3 - 4x^3$$

$$(4y^3 - 12xy^2)y' = 4y^3 - 4x^3$$

$$y' = \frac{4y^3 - 4x^3}{4y^3 - 12xy^2}$$

or $$\frac{4(y^3 - x^3)}{4(y^3 - 3xy^2)} = \frac{y^3 - x^3}{y^3 - 3xy^2}$$

Question 9. Find dy/dx using any correct method.

$$y = \ln\left(\frac{x}{x^2+1}\right)$$

Easiest way: Use rules of logs first

$$y = \ln x - \ln(x^2+1)$$

$$\frac{dy}{dx} = \frac{1}{x} - \frac{1}{x^2+1} \cdot 2x \qquad \text{(CHAIN RULE)}$$

Other method:

$$\frac{dy}{dx} = \frac{1}{\frac{x}{x^2+1}} \cdot \left(\frac{x}{x^2+1}\right)'$$

$$= \frac{x^2+1}{x} \cdot \frac{(x)'(x^2+1) - (x)(x^2+1)'}{(x^2+1)^2}$$

$$= \frac{x^2+1}{x} \cdot \frac{x^2+1 - x \cdot 2x}{(x^2+1)^2} = \frac{x^2+1}{x} \cdot \frac{1-x^2}{(x^2+1)^2}$$

$$= \frac{1-x^2}{x(x^2+1)}$$

Question 10. Find dy/dx using any correct method.

$$y = \ln\sqrt{\frac{x-1}{x+1}}$$

Easiest way: Use rules of logs first.

$$y = \ln\left(\left(\frac{x-1}{x+1}\right)^{1/2}\right) = \frac{1}{2}\ln\left(\frac{x-1}{x+1}\right)$$

$$y = \frac{1}{2}\left(\ln(x-1) - \ln(x+1)\right)$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{x-1}\cdot 1 - \frac{1}{x+1}\cdot 1\right) = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$$

Other possible methods:

$y = \ln u$ and $u = \sqrt{\frac{x-1}{x+1}}$

$u = \sqrt{v} = v^{1/2}$ and $v = \frac{x-1}{x+1}$

$$\frac{dy}{du} = \frac{1}{u}$$

$$\frac{du}{dv} = \frac{1}{2}v^{-1/2}$$

$$\frac{dv}{dx} = \frac{1(x+1)-(x-1)1}{(x+1)^2} = \frac{2}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}$$

$$= \frac{1}{u}\cdot\frac{1}{2}v^{-1/2}\cdot\frac{2}{(x+1)^2}$$

$$= \sqrt{\frac{x+1}{x-1}}\cdot\frac{1}{2}\sqrt{\frac{x+1}{x-1}}\cdot\frac{2}{(x+1)^2} = \frac{x+1}{x-1}\cdot\frac{1}{(x+1)^2} = \frac{1}{x^2-1}$$