WRITE YOUR NAME:

MAC 2311 Section U08 Test 3
Friday April 7th
Total possible score: 20 points (2 points per page)

Question 1. Find $f'(x)$ using any correct method.

$$f(x) = x^5 e^{4x^3}$$

METHOD 1: Product rule $(uv)' = u'v + uv'$

$f'(x) = (x^5)' e^{4x^3} + x^5 (e^{4x^3})'$

$= 5x^4 e^{4x^3} + x^5 e^{4x^3} \cdot 12x^2$

$(e^u)' = e^u \cdot u'$
Chain rule

or $5x^4 e^{4x^3} + 12x^7 e^{4x^3}$ or $(5x^4 + 12x^7) e^{4x^3}$

METHOD 2: $y = x^5 e^{4x^3} \Rightarrow \ln y = \ln(x^5 e^{4x^3})$

$= \ln(x^5) + \ln(e^{4x^3}) = 5 \ln x + 4x^3$

$\Longrightarrow \frac{d}{dx}(\ln y) = \frac{d}{dx}(5 \ln x + 4x^3)$

$\frac{1}{y} \cdot y' = \frac{5}{x} + 12x^2$

$y' = y \cdot \left(\frac{5}{x} + 12x^2\right) = x^5 e^{4x^3}\left(\frac{5}{x} + 12x^2\right)$

$= e^{4x^3}(5x^4 + 12x^7)$

$$\ln(a^r) = r\ln a$$

Question 2. Find dy/dx using any correct method.

$$y = x^{2\tan x}$$

$$\ln y = \ln\left(x^{2\tan x}\right) = 2\tan x \ln x$$

$$\frac{d}{dx}\left(\ln y\right) = \frac{d}{dx}\left(2\tan x \ln x\right)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = 2\left((\tan x)'\ln x + \tan x(\ln x)'\right)$$

$$= 2\left(\sec^2 x \ln x + \tan x \cdot \frac{1}{x}\right)$$

$$\frac{dy}{dx} = 2y\left(\sec^2 x \ln x + \frac{\tan x}{x}\right)$$

$$\text{or } 2x^{2\tan x}\left(\sec^2 x \ln x + \frac{\tan x}{x}\right)$$

Question 3. Find dy/dx if y and x are related by the following equation.

$$x^3 + y^3 = 2xy^2$$

$$\frac{d}{dx}\left(x^3+y^3\right) = \frac{d}{dx}\left(2xy^2\right)$$

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} = (2x)'y^2 + 2x\left(y^2\right)'$$

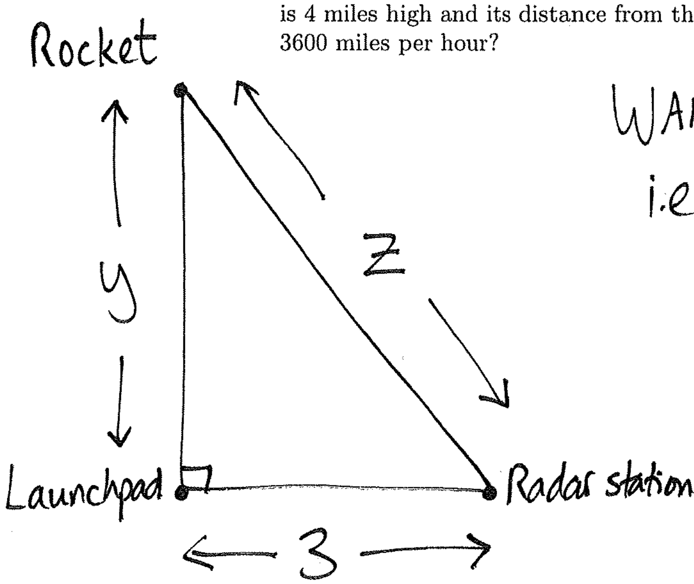
$$3x^2 + 3y^2 \cdot \frac{dy}{dx} = 2y^2 + 2x \cdot 2y \cdot \frac{dy}{dx}$$

$$3y^2\frac{dy}{dx} - 4xy\frac{dy}{dx} = 2y^2 - 3x^2$$

$$\left(3y^2 - 4xy\right)\frac{dy}{dx} = 2y^2 - 3x^2$$

$$\frac{dy}{dx} = \frac{2y^2 - 3x^2}{3y^2 - 4xy}$$

Question 4. A rocket, rising vertically, is tracked by a radar station that is on the ground 3 miles from the launchpad. How fast is the rocket rising when it is 4 miles high and its distance from the radar station is increasing at a rate of 3600 miles per hour?



WANT: rate at which y is increasing

i.e. want $\frac{dy}{dt}$ when $y=4$

GIVEN: $\frac{dz}{dt} = 3600$

Right-angled triangle: We know $3^2 + y^2 = z^2$

$$9 + y^2 = z^2$$

$$\frac{d}{dt}(9+y^2) = \frac{d}{dt}(z^2)$$

$$0 + \underset{\substack{\uparrow \\ 4}}{2y} \cdot \underset{\substack{\uparrow \\ \text{WANT}}}{\frac{dy}{dt}} = \underset{\substack{\uparrow \\ ?}}{2z} \cdot \underset{\substack{\uparrow \\ 3600}}{\frac{dz}{dt}}$$

When $y=4$, we have:

$$9 + 4^2 = z^2$$
$$9 + 16 = z^2$$
$$25 = z^2$$
$$5 = z$$

$$\Rightarrow \quad 2 \cdot 4 \cdot \frac{dy}{dt} = 2 \cdot 5 \cdot 3600 \quad \Rightarrow \quad \frac{dy}{dt} = \frac{\cancel{2} \cdot 5 \cdot 3600}{\cancel{2} \cdot 4}$$

$$= 5 \cdot 900 \quad = 4500$$

Question 5. Find the local linear approximation of the function

$$f(x) = (1+x)^{1/4}$$

at $x_0 = 0$, and use it to estimate $1.04^{1/4}$ and $0.92^{1/4}$.

$$f'(x) = \frac{1}{4}(1+x)^{-3/4} \cdot (1+x)' = \frac{1}{4}(1+x)^{-3/4}$$

$$f(x_0) = f(0) = (1+0)^{1/4} = 1^{1/4} = 1$$

$$f'(x_0) = f'(0) = \frac{1}{4}(1+0)^{-3/4} = \frac{1}{4} \cdot 1 = \frac{1}{4}$$

Linear approximation: $f(x_0) + f'(x_0) \cdot (x - x_0)$

$$= 1 + \frac{1}{4} \cdot (x - 0)$$

$$= 1 + \frac{x}{4}$$

$$1.04^{1/4} = (1+0.04)^{1/4} = f(0.04) \approx 1 + \frac{0.04}{4}$$

$$= 1 + 0.01$$

$$= 1.01$$

$$0.92^{1/4} = (1+(-0.08))^{1/4} = f(-0.08) \approx 1 + \frac{-0.08}{4}$$

$$= 1 - 0.02 = 0.98$$

Question 6. Evaluate the limit.

$$\lim_{x\to 0} \frac{1-\cos 5x}{x^2}$$

Notice $1-\cos(5\cdot 0) = 1-\cos 0 = 1-1 = 0$

$0^2 = 0$

Limit is of the form $\frac{0}{0}$ so we can use L'Hopital

$$\text{Limit} = \lim_{x\to 0} \frac{0-(-\sin 5x)\cdot 5}{2x}$$

$$= \lim_{x\to 0} \frac{5\sin 5x}{2x}$$

This limit is <u>again</u> of the form $\frac{0}{0}$ so we can use L'Hopital again

$$\text{Limit} = \lim_{x\to 0} \frac{5\cos 5x \cdot 5}{2} = \lim_{x\to 0} \frac{25\cos 5x}{2}$$

$$= \frac{25\cos 0}{2} = \frac{25\cdot 1}{2} = \frac{25}{2}$$

Question 7. Evaluate the limit.

$$\lim_{x\to\infty} (5+\ln x)^{1/x^3}$$

Let $L = (5+\ln x)^{1/x^3}$

Then $\ln L = \ln\left((5+\ln x)^{1/x^3}\right)$

$= \frac{1}{x^3}\ln(5+\ln x) = \frac{\ln(5+\ln x)}{x^3}$

If $x\to\infty$, then top $\to \ln\infty = \infty$ and bottom $\to \infty$

so $\ln L$ is of the form $\frac{\infty}{\infty}$ and we can use L'Hopital

$$\ln L = \lim_{x\to\infty} \frac{\frac{1}{5+\ln x}\cdot(5+\ln x)'}{3x^2} = \lim_{x\to\infty} \frac{\frac{1}{5+\ln x}\cdot\frac{1}{x}}{3x^2}$$

$$= \lim_{x\to\infty}\left(\frac{1}{5+\ln x}\cdot\frac{1}{x}\div 3x^2\right) = \lim_{x\to\infty}\left(\frac{1}{5+\ln x}\cdot\frac{1}{x}\cdot\frac{1}{3x^2}\right)$$

$= 0$ because it has the form $\frac{1}{\infty}$.

So $\ln L \to 0$, so $L \to e^0 = 1$.

Question 8. Find the intervals on which the function is increasing, decreasing, concave up, or concave down.

$$f(x) = x^{4/3} - x^{1/3}$$

$$f'(x) = \frac{4}{3}x^{1/3} - \frac{1}{3}x^{-2/3} = \frac{4x^{1/3}}{3} - \frac{1}{3x^{2/3}}$$

$$f''(x) = \frac{4}{3}\cdot\frac{1}{3}x^{-2/3} - \frac{1}{3}\cdot\frac{-2}{3}x^{-5/3}$$

$$= \frac{4}{9x^{2/3}} + \frac{2}{9x^{5/3}} = \frac{4x}{9x^{5/3}} + \frac{2}{9x^{5/3}}$$

$$f'(x) = \frac{4x}{3x^{2/3}} - \frac{1}{3x^{2/3}} = \frac{4x-1}{3x^{2/3}}$$

$$f''(x) = \frac{4x+2}{9x^{5/3}} = \frac{2(2x+1)}{9x^{5/3}}$$

	$x=0$		$x=1/4$	
$4x-1$	− − − − −	− − − − −	0	+ + + + + + + + +
$x^{2/3}$	+ + + + + + + +	0	+ + + + + + +	+ + + + + + +
$f'=\frac{4x-1}{3x^{2/3}}$	− − − − − − −	↑ undef	− − − − −	0 + + + + + + + + +
	f decr		f decr	f incr

	$x=-1/2$		$x=0$	
$2x+1$	− − − − − − − −	0	+ + + + + + + +	+ + + + + + + + + + + +
$x^{5/3}$	− − − − − − − − −	− − − − − − −	0	+ + + + + + + + + +
f''	+ + + + + + + + + + + +	0	− − − − − −	+ + + + + + + + + +
	f conc UP		f conc DOWN	f conc UP

Question 9. Find the intervals where the function is increasing or decreasing.

$$f(x) = \frac{x+1}{x^2+35}$$

$$f'(x) = \frac{(x+1)'(x^2+35) - (x+1)(x^2+35)'}{(x^2+35)^2}$$

$$= \frac{1(x^2+35) - (x+1)2x}{(x^2+35)^2} = \frac{x^2+35-2x^2-2x}{(x^2+35)^2}$$

$$= \frac{-x^2-2x+35}{(x^2+35)^2} = \frac{-(x^2+2x-35)}{(x^2+35)^2}$$

$$= \frac{-(x+7)(x-5)}{(x^2+35)^2}$$

Top is 0 if $x=-7$ or $x=5$

Bottom is never 0

		$x=-7$		$x=5$	
$x+7$	$- - - - - - -$	0	$+ + + + + + + + +$		$+ + + + + + +$
$x-5$	$- - - - - - -$		$- - - - - - - - -$	0	$+ + + + + + +$
f'	$\frac{(-1)(x+7)(x-5)}{(x^2+35)^2}$: $\frac{\text{neg}\cdot\text{neg}\cdot\text{neg}}{\text{pos}}$		$\frac{(-1)(x+7)(x-5)}{(x^2+35)^2}$: $\frac{\text{neg}\cdot\text{pos}\cdot\text{neg}}{\text{pos}}$		$\frac{(-1)(x+7)(x-5)}{(x^2+35)^2}$: $\frac{\text{neg}\cdot\text{pos}\cdot\text{pos}}{\text{pos}}$
	f' neg $\Rightarrow f$ decr		f' pos $\Rightarrow f$ incr		f' neg $\Rightarrow f$ decr

Question 10. Find all the relative extrema of the function, and classify each of them as a minimum or a maximum.

$$f(x) = 3x^4 - 20x^3 + 24x^2$$

$$f'(x) = 12x^3 - 60x^2 + 48x$$
$$= 12x(x^2 - 5x + 4)$$
$$= 12x(x-1)(x-4)$$

When might f' change sign? When $x=0$, $x=1$, or $x=4$

$x=0$ $\qquad$ $x=1$ $\qquad$ $x=4$

x $\quad$ – – – – 0 + + + + + + + + + + + + + + + +

$x-1$ $\quad$ – – – – – – – – – 0 + + + + + + + + + + +

$x-4$ $\quad$ – – – – – – – – – – – – – – – 0 + + + + +

$' = 12x(x-1)(x-4)$ $\quad$ – – – – 0 + + + + + + 0 – – – – – – 0 + + + + +

f' neg	f' pos	f' neg	f' pos
f decr	f incr	f decr	f incr
↘	↗	↘	↗

↑ MIN $\qquad$ ↑ MAX $\qquad$ ↑ MIN