

MAC2311 Section U10  
Suggested problems for Test 1  
(Test 1 is Wednesday October 4th, in class)

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1. Evaluate the limit.

$$\lim_{x \rightarrow 0} \frac{6x - 9}{x^3 - 12x + 3}$$

*Nothing prevents us from plugging in  $x=0$*

$$\frac{0 - 9}{0 - 0 + 3} = \frac{-9}{3} = -3$$

2. Evaluate the limit.

$$\lim_{x \rightarrow 1} \frac{x^4 - 1}{x - 1}$$

*Difference of Squares*

$$\lim_{x \rightarrow 1} \frac{(x^2+1)(x^2-1)}{x-1} = \lim_{x \rightarrow 1} \frac{(x^2+1)(x+1)(x-1)}{x-1} = \lim_{x \rightarrow 1} (x^2+1)(x+1) = (1+1)(1+1) = 4$$

3. Evaluate the limit.

$$\lim_{x \rightarrow 2} \frac{x^2 - 4x + 4}{x^2 + x - 6}$$

*Can factor by trial and error*

$$\lim_{x \rightarrow 2} \frac{(x-2)(x-2)}{(x+3)(x-2)} = \lim_{x \rightarrow 2} \frac{x-2}{x+3} = \frac{2-2}{2+3} = \frac{0}{5} = 0$$

4. Evaluate the limit.

$$\lim_{x \rightarrow 3^+} \frac{x}{x-3} \quad \frac{\text{NONZERO}}{\text{ZERO}}$$

Top approaches 3, which is positive  
Bottom approaches 0 and is positive

$$\begin{aligned} x &\rightarrow 3^+ \\ x &> 3 \\ x-3 &> 0 \end{aligned}$$

Answer:  $+\infty$  or "does not exist"

5. Evaluate the limit.

$$\lim_{x \rightarrow 3^-} \frac{x}{x-3} \quad \frac{\text{NONZERO}}{\text{ZERO}}$$

Top approaches 3, which is positive  
Bottom approaches 0 and is negative

$$\begin{aligned} x &\rightarrow 3^- \\ x &< 3 \\ x-3 &< 0 \end{aligned}$$

Answer:  $-\infty$  or "does not exist"

6. Evaluate the limit.

$$\lim_{x \rightarrow 2^+} \frac{x}{x^2-4} = \lim_{x \rightarrow 2^+} \frac{x}{(x-2)(x+2)} \quad \frac{\text{NONZERO}}{\text{ZERO}}$$

When  $x \rightarrow 2^+$ ,  $x$  is positive and approaches 2  
 $x-2$  is positive and approaches 0  
 $x+2$  is positive and approaches 4

Answer:  $+\infty$   
or "does not exist"

7. Evaluate the limit.

$$\lim_{x \rightarrow 2^-} \frac{x}{x^2-4} = \lim_{x \rightarrow 2^-} \frac{x}{(x-2)(x+2)} \quad \frac{\text{NONZERO}}{\text{ZERO}}$$

When  $x \rightarrow 2^-$ ,  $x$  is positive and approaches 2  
 $x-2$  is negative and approaches 0  
 $x+2$  is positive and approaches 4

So whole thing is  $\frac{\text{POS}}{\text{NEG} \cdot \text{POS}} = \text{NEG}$  and also  $\frac{\text{NONZERO}}{\text{ZERO}}$

Answer:  $-\infty$  or "does not exist"

$\frac{0}{0}$   $\rightarrow$  don't know yet...

8. Evaluate the limit.

$$\lim_{x \rightarrow 9} \frac{x-9}{\sqrt{x}-3}$$

Try multiplying by "conjugate"

$$\begin{aligned} \lim_{x \rightarrow 9} \frac{(x-9)(\sqrt{x}+3)}{(\sqrt{x}-3)(\sqrt{x}+3)} &= \lim_{x \rightarrow 9} \frac{(x-9)(\sqrt{x}+3)}{(\sqrt{x})^2 - 3^2} \\ &= \lim_{x \rightarrow 9} \frac{(x-9)(\sqrt{x}+3)}{x-9} = \lim_{x \rightarrow 9} (\sqrt{x}+3) = \sqrt{9}+3 \\ &= 3+3 = 6 \end{aligned}$$

9. Evaluate the limit.

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+4}-2}{x}$$

$\frac{0}{0}$  Don't know yet...

Conjugate?

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{(\sqrt{x+4}-2)(\sqrt{x+4}+2)}{x(\sqrt{x+4}+2)} &= \lim_{x \rightarrow 0} \frac{(x+4)-4}{x(\sqrt{x+4}+2)} \\ &= \lim_{x \rightarrow 0} \frac{x}{x(\sqrt{x+4}+2)} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+4}+2} = \frac{1}{\sqrt{0+4}+2} \end{aligned}$$

10. Evaluate the limit.

$$\lim_{x \rightarrow \infty} \frac{3x+1}{2x-5}$$

$$= \frac{1}{\sqrt{4}+2} = \frac{1}{2+2} = \frac{1}{4}$$

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{(3x+1) \cdot \frac{1}{x}}{(2x-5) \cdot \frac{1}{x}} &= \lim_{x \rightarrow \infty} \frac{3 + \frac{1}{x}}{2 - \frac{5}{x}} \\ &= \frac{3+0}{2-0} = \frac{3}{2} \end{aligned}$$

11. Evaluate the limit.

$$\lim_{y \rightarrow -\infty} \frac{3}{y+4}$$

Finite  
Infinite

0

12. Evaluate the limit.

$$\lim_{x \rightarrow -\infty} \frac{x-2}{x^2+2x+1}$$

$$\lim_{x \rightarrow -\infty} \frac{(x-2) \cdot \frac{1}{x^2}}{(x^2+2x+1) \cdot \frac{1}{x^2}} = \lim_{x \rightarrow -\infty} \frac{\frac{1}{x} - \frac{2}{x^2}}{1 + \frac{2}{x} + \frac{1}{x^2}}$$

13. Evaluate the limit.

$$\lim_{x \rightarrow \infty} \frac{5x^2+7}{3x^2-x}$$

$$\lim_{x \rightarrow \infty} \frac{(5x^2+7) \cdot \frac{1}{x^2}}{(3x^2-x) \cdot \frac{1}{x^2}} = \lim_{x \rightarrow \infty} \frac{5 + \frac{7}{x^2}}{3 - \frac{1}{x}} = \frac{5+0}{3-0} = \frac{5}{3}$$

$$= \frac{0-0}{1+0+0} = \frac{0}{1} = 0$$

14. Evaluate the limit.

$$\lim_{t \rightarrow -\infty} \frac{5-2t^3}{t^2+1}$$

$$\lim_{t \rightarrow -\infty} \frac{(5-2t^3) \cdot \frac{1}{t^3}}{(t^2+1) \cdot \frac{1}{t^3}} = \lim_{t \rightarrow -\infty} \frac{\frac{5}{t^3} - 2}{\frac{1}{t} + \frac{1}{t^3}}$$

Approaches  $\frac{0-2}{0+0} = \frac{-2}{0}$  Nonzero / Zero Limit does not exist.

Also, if  $t \rightarrow -\infty$ , then  $\frac{5-2t^3}{t^2+1} \approx \frac{-2t^3}{t^2} = -2t \rightarrow +\infty$

15. Evaluate the limit.

$$\lim_{x \rightarrow -\infty} \frac{x + 4x^3}{1 - x^2 + 7x^3}$$

$$\lim_{x \rightarrow -\infty} \frac{(x + 4x^3) \cdot \frac{1}{x^3}}{(1 - x^2 + 7x^3) \cdot \frac{1}{x^3}} = \lim_{x \rightarrow -\infty} \frac{\frac{1}{x^2} + 4}{\frac{1}{x^3} - \frac{1}{x} + 7}$$

$$= \frac{0 + 4}{0 - 0 + 7} = \frac{4}{7}$$

16. Evaluate the limit.

$$\lim_{x \rightarrow \infty} \sqrt{\frac{2 - 3x + 4x^2}{1 + 9x^2}}$$

$$\lim_{x \rightarrow \infty} \sqrt{\frac{(2 - 3x + 4x^2) \cdot \frac{1}{x^2}}{(1 + 9x^2) \cdot \frac{1}{x^2}}} = \lim_{x \rightarrow \infty} \sqrt{\frac{\frac{2}{x^2} - \frac{3}{x} + 4}{\frac{1}{x^2} + 9}}$$

$$= \sqrt{\frac{0 - 0 + 4}{0 + 9}} = \sqrt{\frac{4}{9}} = \frac{2}{3}$$

17. Evaluate the limit.

$$\lim_{x \rightarrow \infty} \frac{\sqrt{5x^2 - 2}}{x + 3}$$

Idea: Top "really" grows like  $\sqrt{x^2} = x$

$$\lim_{x \rightarrow \infty} \frac{\sqrt{5x^2 - 2} \cdot \frac{1}{x}}{(x + 3) \cdot \frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{\sqrt{5x^2 - 2} \cdot \sqrt{\frac{1}{x^2}}}{(x + 3) \cdot \frac{1}{x}}$$

$$= \lim_{x \rightarrow \infty} \frac{\sqrt{(5x^2 - 2) \cdot \frac{1}{x^2}}}{(x + 3) \cdot \frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{\sqrt{5 - \frac{2}{x^2}}}{1 + \frac{3}{x}}$$

$$= \frac{\sqrt{5 - 0}}{1 + 0} = \frac{\sqrt{5}}{1} = \sqrt{5}$$

18. Find all values of  $x$ , if any, at which  $f$  is not continuous.

$$f(x) = (x-8)^{1/3}$$

Continuous for all  $x$ .

(Composition of  $x-8$ , which is continuous everywhere,  
and cube root function, which is continuous everywhere.)

19. Find all values of  $x$ , if any, at which  $f$  is not continuous.

$$f(x) = \frac{x+2}{x^2-4}$$

Discontinuous when  $x^2-4=0$      $x^2=4$      $x=\pm 2$

(Alternatively:  $x^2-4=0 \Rightarrow (x-2)(x+2)=0 \Rightarrow x=2$  or  $-2$ )

20. Find all values of  $x$ , if any, at which  $f$  is not continuous.

$$f(x) = \frac{x}{2x^2+x}$$

Discontinuous when  $2x^2+x=0$

$$x(2x+1)=0$$

$x=0$  or  $2x+1=0 \Rightarrow 2x=-1 \Rightarrow x=-\frac{1}{2}$

21. Find all values of  $x$ , if any, at which  $f$  is not continuous.

$$f(x) = \frac{3}{x} + \frac{x-1}{x^2-1}$$

Discontinuous when  $x=0$  or  $x^2-1=0$

$$x^2=1$$

$$x=1 \text{ or } -1$$

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Discontinuous if  $x=0$  or  $1$  or  $-1$

22. Find all values of  $x$  (if any) at which  $f$  is not continuous, and determine whether each discontinuity is a removable discontinuity.

$$\lim_{x \rightarrow -3} \frac{x^2 + 3x}{x + 3} = \lim_{x \rightarrow -3} \frac{x(x+3)}{x+3} = \lim_{x \rightarrow -3} x = -3$$

$f(x) = \frac{x^2 + 3x}{x + 3}$  Discontinuous at  $x = -3$ .  
Limit exists  
Removable.

23. Find the discontinuities, if any.

$$f(x) = \sin(x^2 - 2)$$

No discontinuities. Composition of polynomial  $x^2 - 2$ , which is continuous everywhere, and sine function, which is continuous everywhere.

24. Find the discontinuities, if any.

$$f(x) = \cos\left(\frac{x}{x - \pi}\right)$$

Cosine function is continuous everywhere, but  $\frac{x}{x - \pi}$  is discontinuous when  $x = \pi$ .

25. Find the discontinuities, if any.

$$f(x) = |\cot x| \quad \text{Recall } \cot x = \frac{\cos x}{\sin x}$$

Discontinuous when  $\sin x = 0$ , i.e. when  $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \dots$

26. Find the discontinuities, if any.

$$f(x) = \frac{1}{1 + \sin^2 x}$$

No discontinuities. Discontinuous if  $1 + \sin^2 x = 0$

i.e. if  $\sin^2 x = -1$

$$\sin x = \sqrt{-1} \quad \text{Undefined if using real numbers}$$

Never happens

27. Find the limit.

$$\lim_{x \rightarrow \infty} \cos\left(\frac{1}{x}\right)$$

Since  $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$ , we conclude  $\lim_{x \rightarrow \infty} \cos\left(\frac{1}{x}\right)$   
 $= \cos 0 = 1$

28. Find the limit.

$$\lim_{x \rightarrow \infty} \sin\left(\frac{\pi x}{2-3x}\right)$$

$$\begin{aligned} \lim_{x \rightarrow \infty} \sin\left(\frac{\pi x \cdot \frac{1}{x}}{(2-3x) \cdot \frac{1}{x}}\right) &= \lim_{x \rightarrow \infty} \sin\left(\frac{\pi}{\frac{2}{x} - 3}\right) \\ &= \sin\left(\frac{\pi}{0-3}\right) = \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} \end{aligned}$$

29. Find the limit.

$$\lim_{x \rightarrow \infty} \sin^{-1}\left(\frac{x}{1-2x}\right)$$

$$\begin{aligned} \lim_{x \rightarrow \infty} \sin^{-1}\left(\frac{x \cdot \frac{1}{x}}{(1-2x) \cdot \frac{1}{x}}\right) &= \lim_{x \rightarrow \infty} \sin^{-1}\left(\frac{1}{\frac{1}{x} - 2}\right) \\ &= \sin^{-1}\left(\frac{1}{0-2}\right) = \sin^{-1}\left(-\frac{1}{2}\right) \end{aligned}$$

30. Find the limit.

$$\lim_{x \rightarrow \infty} \ln\left(\frac{x+1}{x}\right)$$

$$\begin{aligned} \lim_{x \rightarrow \infty} \ln\left(\frac{(x+1) \cdot \frac{1}{x}}{x \cdot \frac{1}{x}}\right) &= \lim_{x \rightarrow \infty} \ln\left(\frac{1 + \frac{1}{x}}{1}\right) \\ &= \ln\left(\frac{1+0}{1}\right) = \ln\left(\frac{1}{1}\right) = \ln 1 = 0 \end{aligned}$$



31. Write down the definition of the derivative.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

32. Use the definition of the derivative to find the derivative of  $f(x) = x^2$ .

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} = \lim_{h \rightarrow 0} \frac{(2x+h)h}{h} = \lim_{h \rightarrow 0} (2x+h) \\ &= 2x + 0 = 2x \end{aligned}$$

33. Use the definition of the derivative to find the derivative of  $f(x) = \frac{1}{x}$ .

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \cdot \frac{(x+h)x}{(x+h)x} \\ &= \lim_{h \rightarrow 0} \frac{x - (x+h)}{h(x+h)x} = \lim_{h \rightarrow 0} \frac{-h}{h(x+h)x} \\ &= \lim_{h \rightarrow 0} \frac{-1}{(x+h)x} = \frac{-1}{(x+0)x} = \frac{-1}{x^2} \end{aligned}$$