WRITE YOUR NAME:

MAC 2311 Section U10 Test 2
Wednesday October 25th
Total possible score: 20 points (2 points per page)

Question 1. Find dy/dx.

$$y = 4x^7 + 2x^3 - 89$$

$$\frac{dy}{dx} = 4 \cdot 7x^6 + 2 \cdot 3x^2 + 0$$

$$= 28x^6 + 6x^2$$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

$(a+b)^2 = (a+b)(a+b) = aa + ab + ba + bb = a^2 + 2ab + b^2$

$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$

or $(a+b)^3 = (a+b)^2(a+b) = (a^2 + 2ab + b^2)(a+b)$

$= (a^2+2ab+b^2)\cdot a + (a^2+2ab+b^2)\cdot b$ etc.

Question 2a. Write down the **definition** of the derivative.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Question 2b. Use the **definition** of the derivative to find the derivative of $f(x) = x^3$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h\to 0} \frac{(3x^2 + 3xh + h^2)h}{h}$$

$$= \lim_{h\to 0} (3x^2 + 3xh + h^2)$$

$$= 3x^2 + 0 + 0 = 3x^2$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Question 3. Find $f'(x)$, and simplify.

$$f(x) = \frac{2x^{3/2} + 1}{2x^{1/2} + 1}$$

$$f'(x) = \frac{(2x^{3/2}+1)'(2x^{1/2}+1) - (2x^{3/2}+1)(2x^{1/2}+1)'}{(2x^{1/2}+1)^2}$$

$$= \frac{\left(2\cdot\frac{3}{2}x^{1/2}+0\right)(2x^{1/2}+1) - (2x^{3/2}+1)\left(2\cdot\frac{1}{2}x^{-1/2}+0\right)}{(2x^{1/2}+1)^2}$$

$$= \frac{3x^{1/2}(2x^{1/2}+1) - (2x^{3/2}+1)x^{-1/2}}{(2x^{1/2}+1)^2}$$

$$= \frac{6x + 3x^{1/2} - 2x - x^{-1/2}}{(2x^{1/2}+1)^2}$$

$$= \frac{4x + 3x^{1/2} - x^{-1/2}}{(2x^{1/2}+1)^2}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Question 4. Find the second derivative $f''(x)$.

$$f(x) = \frac{3x+7}{2x+5}$$

$$f'(x) = \frac{(3x+7)'(2x+5) - (3x+7)(2x+5)'}{(2x+5)^2}$$

$$= \frac{3\cdot(2x+5) - (3x+7)\cdot 2}{(2x+5)^2} = \frac{(6x+15) - (6x+14)}{(2x+5)^2}$$

$$= \frac{6x+15-6x-14}{(2x+5)^2} = \frac{1}{(2x+5)^2} \quad \text{OR} \quad (2x+5)^{-2}$$

So $f'(x) = (2x+5)^{-2}$ is of the form u^{-2} or $(g(x))^{-2}$ and we can use the chain rule

$$f''(x) = -2(2x+5)^{-3}\cdot(2x+5)'$$

$$= -2(2x+5)^{-3}\cdot 2$$

or $-4(2x+5)^{-3}$

or $\dfrac{-4}{(2x+5)^3}$

Question 5. Find $f'(x)$ and simplify.

$$f(x) = \frac{5 - \cos x}{5 + \sin x}$$

$$f'(x) = \frac{(5-\cos x)'(5+\sin x) - (5-\cos x)(5+\sin x)'}{(5+\sin x)^2}$$

$$= \frac{(0-(-\sin x))(5+\sin x) - (5-\cos x)(0+\cos x)}{(5+\sin x)^2}$$

$$= \frac{\sin x\,(5+\sin x) - (5-\cos x)\cos x}{(5+\sin x)^2}$$

$$= \frac{(5\sin x + \sin^2 x) - (5\cos x - \cos^2 x)}{(5+\sin x)^2}$$

$$= \frac{5\sin x + \sin^2 x - 5\cos x + \cos^2 x}{(5+\sin x)^2}$$

$$= \frac{5\sin x - 5\cos x + 1}{(5+\sin x)^2}$$

Question 6. Find $f'(x)$ using any correct method.

$$f(x) = \left(\frac{x^2-1}{x^2+1}\right)^{17}$$

METHOD 1: $f = u^{17}$ and $u = \dfrac{x^2-1}{x^2+1}$

So $f'(x) = \dfrac{df}{dx} = \dfrac{df}{du} \cdot \dfrac{du}{dx} = 17u^{16} \cdot \dfrac{du}{dx}$

$$= 17\left(\frac{x^2-1}{x^2+1}\right)^{16} \cdot \left(\frac{x^2-1}{x^2+1}\right)'$$

$$= 17\left(\frac{x^2-1}{x^2+1}\right)^{16} \cdot \frac{2x\cdot(x^2+1)-(x^2-1)\cdot 2x}{(x^2+1)^2}$$

$$= 17\left(\frac{x^2-1}{x^2+1}\right)^{16} \cdot \frac{2x^3+2x-2x^3+2x}{(x^2+1)^2} = 17\left(\frac{x^2-1}{x^2+1}\right)^{16} \cdot \frac{4x}{(x^2+1)^2}$$

METHOD 2: $\ln f(x) = \ln\left(\left(\dfrac{x^2-1}{x^2+1}\right)^{17}\right) = 17\ln\left(\dfrac{x^2-1}{x^2+1}\right)$

$= 17\left(\ln(x^2-1) - \ln(x^2+1)\right)$. Then take $\dfrac{d}{dx}$ of each side

$$\Rightarrow \frac{1}{f(x)} \cdot f'(x) = 17\left(\frac{1}{x^2-1}\cdot 2x - \frac{1}{x^2+1}\cdot 2x\right)$$

$$f'(x) = 17f(x)\cdot\left(\frac{2x}{x^2-1} - \frac{2x}{x^2+1}\right)$$

which can be rewritten in various ways

Question 7. Find $f'(x)$.

$$f(x) = \cos^3\left(\frac{x}{x+1}\right)$$

or equivalently $$f(x) = \left(\cos\left(\frac{x}{x+1}\right)\right)^3$$

$f = (\text{something})^3$

$f = u^3$ and $u = \cos\left(\frac{x}{x+1}\right)$

$\downarrow$

$u = \cos v$ and $v = \frac{x}{x+1}$

$\frac{df}{du} = 3u^2$ $\quad \frac{du}{dv} = -\sin v$ $\quad \frac{dv}{dx} = \frac{(x)'(x+1)-(x)(x+1)'}{(x+1)^2}$

$= \frac{1(x+1)-x\cdot 1}{(x+1)^2} = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2}$

$f'(x) = \frac{df}{dx} = \frac{df}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}$

$= 3u^2\cdot(-\sin v)\cdot\frac{1}{(x+1)^2}$

$= 3\cos^2\left(\frac{x}{x+1}\right)\cdot\left(-\sin\frac{x}{x+1}\right)\cdot\frac{1}{(x+1)^2}$

Question 8. Find dy/dx using any correct method.

$$x^4 + y^4 = 4xy^3$$

$$\frac{d}{dx}\left(x^4 + y^4\right) = \frac{d}{dx}\left(4x \cdot y^3\right)$$

Product rule

$$\frac{d}{dx}\left(x^4\right) + \frac{d}{dx}\left(y^4\right) = (4x)' \cdot y^3 + 4x \cdot \left(y^3\right)'$$

where prime means $\frac{d}{dx}$

chain rule

$$4x^3 + 4y^3 \cdot y' = 4 \cdot y^3 + 4x \cdot 3y^2 \cdot y'$$

y is a FUNCTION of x

chain rule

$$4x^3 + 4y^3 \cdot y' = 4y^3 + 12xy^2 \cdot y'$$

$$-4x^3 \quad -12xy^2 \cdot y' \qquad -4x^3 \quad -12xy^2 \cdot y'$$

$$4y^3 \cdot y' - 12xy^2 \cdot y' = 4y^3 - 4x^3$$

$$\left(4y^3 - 12xy^2\right)y' = 4y^3 - 4x^3$$

$$y' = \frac{4y^3 - 4x^3}{4y^3 - 12xy^2} = \frac{4\left(y^3 - x^3\right)}{4\left(y^3 - 3xy^2\right)} = \frac{y^3 - x^3}{y^3 - 3xy^2}$$

Question 9. Find dy/dx using any correct method.

$$y = \ln\left(\frac{x}{x^2+1}\right)$$

METHOD 1: $y = \ln u$ and $u = \dfrac{x}{x^2+1}$

$$\frac{dy}{du} = \frac{1}{u} \qquad \frac{du}{dx} = \frac{(x)'(x^2+1)-(x)(x^2+1)'}{(x^2+1)^2}$$

$$= \frac{1(x^2+1)-x\cdot 2x}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2}$$

$$= \frac{1-x^2}{(x^2+1)^2}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{u}\cdot\frac{1-x^2}{(x^2+1)^2}$$

$$= \frac{x^2+1}{x}\cdot\frac{1-x^2}{(x^2+1)^2} = \frac{1-x^2}{x(x^2+1)}$$

METHOD 2: $y = \ln\left(\dfrac{x}{x^2+1}\right) = \ln x - \ln(x^2+1)$

$$\frac{dy}{dx} = \frac{1}{x} - \frac{1}{x^2+1}\cdot(x^2+1)' = \frac{1}{x} - \frac{2x}{x^2+1}$$

which can also be written $\dfrac{x^2+1}{x(x^2+1)} - \dfrac{2x^2}{x(x^2+1)}$

or $\dfrac{1-x^2}{x(x^2+1)}$

Question 10. Find dy/dx using any correct method.

$$y = \ln\sqrt{\frac{x-1}{x+1}}$$

METHOD 1: $y = \ln u$ and $u = \sqrt{\frac{x-1}{x+1}} \Rightarrow u = \sqrt{v} = v^{1/2}$ and $v = \frac{x-1}{x+1}$

$$\frac{dy}{du} = \frac{1}{u}$$

$$\frac{du}{dv} = \frac{1}{2}v^{-1/2}$$

or $\frac{1}{2}\cdot\frac{1}{\sqrt{v}}$

$$\frac{dv}{dx} = \frac{(x-1)'(x+1)-(x-1)(x+1)'}{(x+1)^2}$$

$$= \frac{1\cdot(x+1)-(x-1)\cdot 1}{(x+1)^2} = \frac{2}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx} = \frac{1}{u}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{v}}\cdot\frac{2}{(x+1)^2}$$

$$= \sqrt{\frac{x+1}{x-1}}\cdot\frac{1}{2}\cdot\sqrt{\frac{x+1}{x-1}}\cdot\frac{2}{(x+1)^2} = \frac{x+1}{x-1}\cdot\frac{1}{(x+1)^2}$$

$$= \frac{1}{(x-1)(x+1)}$$

METHOD 2: $y = \ln\left(\left(\frac{x-1}{x+1}\right)^{1/2}\right)$

$$= \frac{1}{2}\ln\left(\frac{x-1}{x+1}\right) = \frac{1}{2}\Big(\ln(x-1) - \ln(x+1)\Big).$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{x-1}\cdot 1 - \frac{1}{x+1}\cdot 1\right) = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$$

$$= \frac{1}{2}\left(\frac{x+1}{(x-1)(x+1)} - \frac{x-1}{(x+1)(x-1)}\right) = \frac{1}{2}\left(\frac{2}{(x-1)(x+1)}\right)$$

$$= \frac{1}{(x-1)(x+1)}$$