MAC2311 Section U10
Suggested problems for Test 2
(Test 2 is Wednesday October 25th, in class)

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1. Write down the **definition** of the derivative.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

2. Use the **definition** of the derivative to find the derivative of $f(x) = x^2$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0} \frac{2xh+h^2}{h} = \lim_{h\to 0} \frac{(2x+h)h}{h}$$

$$= \lim_{h\to 0}(2x+h)$$

$$= 2x+0 = 2x$$

3. Use the **definition** of the derivative to find the derivative of $f(x) = \frac{1}{x}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \cdot \frac{x(x+h)}{x(x+h)}$$

$$= \lim_{h\to 0} \frac{x-(x+h)}{hx(x+h)} = \lim_{h\to 0} \frac{x-x-h}{hx(x+h)} = \lim_{h\to 0} \frac{-h}{hx(x+h)}$$

$$= \lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x(x+0)} = \frac{-1}{x^2}.$$

4. Find dy/dx.

$$y = 3x^8 + 2x + 1$$

$$y' = 3 \cdot 8x^7 + 2 \cdot 1 + 0$$
$$= 24x^7 + 2$$

5. Find dy/dx.

$$y = \frac{1}{2}(x^4 + 7) = \frac{1}{2}x^4 + \frac{7}{2}$$

$$y' = \frac{1}{2} \cdot 4x^3 + 0 = 2x^3$$

6. Find dy/dx.

$\sqrt{2} \cdot x$, not $\sqrt{2x}$

$$y = \sqrt{2}x + (1/\sqrt{2})$$

$$y = \sqrt{2} \cdot x + \frac{1}{\sqrt{2}}$$

$$y' = \sqrt{2} \cdot 1 + 0 = \sqrt{2}$$

7. Find dy/dx.

$$y = \frac{x^2 + 1}{5} = \frac{1}{5}(x^2 + 1) = \frac{1}{5}x^2 + \frac{1}{5}$$

$$y' = \frac{1}{5} \cdot 2x + 0 = \frac{2}{5}x \text{ or } \frac{2x}{5}$$

8. Find $f'(x)$.

$$f(x) = x^{-3} + \frac{1}{x^7} = x^{-3} + x^{-7}$$

$f'(x) = -3x^{-4} - 7x^{-8}$. Can stop there.

Other correct answers: $-\frac{3}{x^4} - \frac{7}{x^8}$ or $\frac{-3x^4-7}{x^8}$

9. Find $f'(x)$.

$$f(x) = \sqrt{x} + \frac{1}{x} = x^{1/2} + x^{-1}$$

$f'(x) = \frac{1}{2}x^{-1/2} - 1x^{-2}$ Can stop there.

Can also write as $\frac{1}{2\sqrt{x}} - \frac{1}{x^2}$

10. Find $f'(x)$.

$$f(x) = x^e + \frac{1}{x^{\sqrt{10}}} = x^e + x^{-\sqrt{10}}$$

Remember, e and $-\sqrt{10}$ are just numbers (constants)

$f'(x) = ex^{e-1} - \sqrt{10}\,x^{-\sqrt{10}-1}$

11. Find $f'(x)$.

$$f(x) = (3x^2+1)^2$$ Deriv. of u^2 will be $2u \cdot u'$

$$f'(x) = 2(3x^2+1)\cdot(3x^2+1)'$$
$$= 2(3x^2+1)\cdot 6x$$ Can stop there

Can also write $f'(x) = 12x(3x^2+1)$ or $36x^3+12x$

ALSO can write $f(x) = 9x^4+6x^2+1$ so $f'(x) = 36x^3+12x$

12. Find $f'(x)$.

$$f(x) = (3x^2+6)(2x-\frac{1}{4})$$

More than one correct method. Can expand first.

$$f(x) = 6x^3 - \frac{3}{4}x^2 + 12x - \frac{6}{4}$$

$$f'(x) = 18x^2 - \frac{6}{4}x + 12 \quad \text{or} \quad 18x^2 - \frac{3}{2}x + 12$$

Can also use product rule.

13. Find $f'(x)$.

$$f(x) = (2-x-3x^3)(7+x^5)$$

More than one correct method. I will use product rule.

$$f'(x) = (2-x-3x^3)'(7+x^5) + (2-x-3x^3)(7+x^5)'$$
$$= (-1-9x^2)(7+x^5) + (2-x-3x^3)\cdot 5x^4$$

Since the question just said find $f'(x)$, you can stop there. However, it can be good practice to simplify.

$$f'(x) = -7 - x^5 - 63x^2 - 9x^7 + 10x^4 - 5x^5 - 15x^7$$
$$= -24x^7 - 6x^5 + 10x^4 - 63x^2 - 7$$

14. Find $f'(x)$.

$$f(x) = \frac{x-2}{x^4+x+1}$$

$$f'(x) = \frac{(x-2)'(x^4+x+1) - (x-2)(x^4+x+1)'}{(x^4+x+1)^2}$$

$$= \frac{x^4+x+1-(x-2)(4x^3+1)}{(x^4+x+1)^2}$$

Can stop there, but can also simplify. Numerator is $x^4+x+1-(4x^4-8x^3+x-2)$ etc.

15. Find $f'(x)$.

$$f(x) = (x^3+2x)^{37}$$

$f = u^{37}$ and $u = x^3+2x$

Deriv. of u^{37} is $37u^{36} \cdot u'$ CHAIN RULE

$$f'(x) = 37(x^3+2x)^{36} \cdot (x^3+2x)'$$

$$= 37(x^3+2x)^{36} \cdot (3x^2+2).$$

16. Find $f'(x)$.

$$f(x) = \sin\left(\frac{1}{x^2}\right) = \sin(x^{-2})$$

Deriv. of $\sin u$ is $\cos u \cdot u'$ CHAIN RULE

$$f'(x) = \cos(x^{-2}) \cdot (x^{-2})'$$

$$= \cos(x^{-2}) \cdot (-2x^{-3})$$

Can stop there

Which can also be written in other ways

$$\cos\left(\frac{1}{x^2}\right) \cdot \frac{-2}{x^3} \quad \text{or} \quad \frac{-2\cos(1/x^2)}{x^3}$$

17. Find $f'(x)$ using any correct method.

$$f(x) = \left(\frac{1+x^2}{1-x^2}\right)^{17}$$

$$f'(x) = 17\left(\frac{1+x^2}{1-x^2}\right)^{16} \cdot \left(\frac{1+x^2}{1-x^2}\right)'$$

Chain rule, then later, quotient rule

$$= 17\left(\frac{1+x^2}{1-x^2}\right)^{16} \cdot \frac{(1+x^2)'(1-x^2) - (1+x^2)(1-x^2)'}{(1-x^2)^2}$$

$$= 17\left(\frac{1+x^2}{1-x^2}\right)^{16} \cdot \frac{2x(1-x^2) - (1+x^2)(-2x)}{(1-x^2)^2}$$

Can stop there, but it might be good practice to simplify the numerator.

18. Find dy/dx using any correct method.

$$x^3 + y^3 = 3xy^2$$

Implicit differentiation. y is "secretly" a function of x.
We must treat y^3 and y^2 like $(g(x))^3$ and $(g(x))^2$.

$$\frac{d}{dx}\left[x^3 + y^3\right] = \frac{d}{dx}\left[3xy^2\right]$$

$$3x^2 + 3y^2 \cdot y' = (3x)'y^2 + 3x(y^2)'$$

where prime means $\frac{d}{dx}$

$$3x^2 + 3y^2 \cdot y' = 3y^2 + 3x \cdot 2y \cdot y'$$

$$3y^2 \cdot y' - 6xy \cdot y' = 3y^2 - 3x^2$$

$$(3y^2 - 6xy)y' = 3y^2 - 3x^2$$

$$y' = \frac{3y^2 - 3x^2}{3y^2 - 6xy} \text{ or } \frac{y^2 - x^2}{y^2 - 2xy} \text{ or } \frac{(y+x)(y-x)}{y(y-2x)}$$

NOTE: Asking for the SECOND derivative

19. Find d^2y/dx^2 using any correct method.

$x^3 + y^3 = 1$ Can use implicit differentiation

$$\frac{d}{dx}\left[x^3+y^3\right] = \frac{d}{dx}\left[1\right] \Rightarrow 3x^2+3y^2y' = 0 \Rightarrow 3y^2y' = -3x^2$$

$$\Rightarrow y' = \frac{-3x^2}{3y^2} = -\frac{x^2}{y^2}. \text{ Then } y'' = -\left(\frac{x^2}{y^2}\right)' = -\frac{(x^2)'y^2 - x^2(y^2)'}{(y^2)^2}$$

$$= -\frac{2xy^2 - x^2 \cdot 2yy'}{y^4} = -\frac{2xy^2 - 2x^2y\left(-\frac{x^2}{y^2}\right)}{y^4}$$ Can stop there, or can simplify.

Maybe easier: Start with $y^3 = 1 - x^3 \Rightarrow y = (1-x^3)^{1/3}$

$$\Rightarrow y' = \frac{1}{3}(1-x^3)^{-2/3} \cdot (-3x^2) = -x^2(1-x^3)^{-2/3}$$

Then take derivative AGAIN. Still time-consuming.

20. Find dy/dx.

$y = \ln 5x$

METHOD 1. $y = \ln(5x)$

$$y' = \frac{1}{5x} \cdot (5x)' = \frac{1}{5x} \cdot 5 = \frac{1}{x}$$

METHOD 2. First rewrite using rules of logs

$$y = \ln(5x) = \underbrace{\ln 5}_{\text{constant}} + \ln x$$

$$y' = 0 + \frac{1}{x} = \frac{1}{x}$$

21. Find dy/dx.

$$y = \ln|x^2 - 1|$$

FACT: Derivative of $\ln|x|$ is $\frac{1}{x}$. Don't need to "do" anything with the absolute value.

$y = \ln|u|$ and $u = x^2 - 1$

$$\frac{dy}{du} = \frac{1}{u} \quad \text{and} \quad \frac{du}{dx} = 2x \Rightarrow \frac{dy}{dx} = \frac{1}{u} \cdot 2x = \frac{2x}{x^2-1}$$

22. Find dy/dx.

$$y = \ln(x^2)$$

METHOD 1. $y' = \frac{1}{x^2} \cdot (x^2)' = \frac{1}{x^2} \cdot 2x = \frac{2}{x}$

METHOD 2. By rules of logs, $y = \ln(x^2) = 2\ln x$

$$y' = 2 \cdot \frac{1}{x} = \frac{2}{x}$$

23. Find dy/dx.

$$y = (\ln x)^3$$

$$y' = 3(\ln x)^2 \cdot (\ln x)'$$

$$= 3(\ln x)^2 \cdot \frac{1}{x} \quad \text{or} \quad \frac{3(\ln x)^2}{x}$$

24. Find dy/dx.

$$y = \ln\left(\frac{x}{x^2+1}\right)$$

More than one correct method, but easiest to use log rules first.

$$y = \ln x - \ln(x^2+1)$$

$$y' = \frac{1}{x} - \frac{1}{x^2+1}\cdot(x^2+1)'$$

$$= \frac{1}{x} - \frac{2x}{x^2+1}.$$

25. Find dy/dx.

$$y = \ln(\ln x)$$

$$y' = \frac{1}{\ln x}\cdot(\ln x)'$$

$$= \frac{1}{\ln x}\cdot\frac{1}{x}$$