WRITE YOUR NAME:

MAC 2311 Section U10 Test 3
Wednesday November 15th
Total possible score: 20 points (2 points per page)

Question 1. Find $f'(x)$ using any correct method.

$$f(x) = x^5 e^{4x^3}$$

METHOD 1: $f'(x) = (x^5)' e^{4x^3} + x^5 \left(e^{4x^3}\right)'$

$= 5x^4 e^{4x^3} + x^5 e^{4x^3} \cdot (4x^3)'$

$= 5x^4 e^{4x^3} + x^5 e^{4x^3} \cdot 12x^2.$

This can also be written $5x^4 e^{4x^3} + 12x^7 e^{4x^3}$

or $(5x^4 + 12x^7) e^{4x^3}$ or $x^4(5+12x^3)e^{4x^3}$.

METHOD 2: $y = x^5 e^{4x^3}$

$\ln y = \ln\left(x^5 e^{4x^3}\right) = \ln(x^5) + \ln\left(e^{4x^3}\right)$

$\ln y = 5 \ln x + 4x^3$

$\frac{d}{dx}(\ln y) = \frac{d}{dx}(5 \ln x + 4x^3)$

$\frac{1}{y} \cdot y' = \frac{5}{x} + 12x^2$

$y' = y \cdot \left(\frac{5}{x} + 12x^2\right) = x^5 e^{4x^3}\left(\frac{5}{x} + 12x^2\right)$

which is the same as
$e^{4x^3}(5x^4 + 12x^7)$

Question 2. Find dy/dx using any correct method.

$$y = x^{2\sin x}$$

$$\ln y = \ln\left(x^{2\sin x}\right)$$

$$\ln y = 2\sin x \cdot \ln(x)$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(2\sin x \cdot \ln x\right)$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2\left((\sin x)' \ln x + \sin x (\ln x)'\right)$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2\left(\cos x \ln x + \sin x \cdot \frac{1}{x}\right)$$

$$\frac{dy}{dx} = 2y\left(\cos x \ln x + \frac{\sin x}{x}\right)$$

You can stop there.
You can also rewrite as

$$\frac{dy}{dx} = 2x^{2\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right)$$

Question 3. Find dy/dx if y and x are related by the following equation.

$$x^3 + y^3 = 2xy^2$$

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}\left(2xy^2\right)$$

$$3x^2 + 3y^2 \cdot y' = (2x)' y^2 + 2x \left(y^2\right)'$$

$$3x^2 + 3y^2 \cdot y' = 2y^2 + 2x \cdot 2y \cdot y'$$

$$3x^2 + 3y^2 \cdot y' = 2y^2 + 4xy \cdot y'$$

$$3y^2 \cdot y' - 4xy \cdot y' = 2y^2 - 3x^2$$

$$\left(3y^2 - 4xy\right) \cdot y' = 2y^2 - 3x^2$$

$$y' = \frac{2y^2 - 3x^2}{3y^2 - 4xy}$$

(can factor out y in bottom)

Question 4. Find dy/dx if y and x are related by the following equation. You may use any **correct** method.

$$x = \tan y$$

METHOD 1: $\frac{d}{dx}(x) = \frac{d}{dx}(\tan y)$

$$1 = \sec^2 y \cdot \frac{dy}{dx}$$

$$\frac{1}{\sec^2 y} = \frac{dy}{dx}$$

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

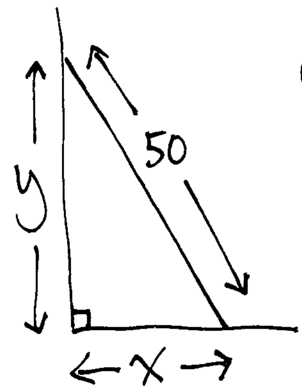
$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{\tan^2 y + 1} = \frac{1}{x^2+1}$$

METHOD 2: $x = \tan y \Rightarrow y = \tan^{-1} x$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2}$$ (can memorize)

Question 5. A 50 ft ladder is leaning against a wall. If the bottom of the ladder is sliding along the ground at 2 ft/sec, how fast will the top of the ladder be sliding down the wall when the bottom is 30 ft away from the wall?

DRAW A PICTURE AND MAKE UP NAMES!



Given: x increases at rate of 2 ft/sec

$$\Rightarrow \frac{dx}{dt} = 2$$

WANT: rate at which y is changing

i.e. WANT $\frac{dy}{dt}$

at the moment when x is 30.

By geometry, we know $x^2 + y^2 = 50^2$.

Then $\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(50^2)$

$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$$

NOTE: When $x=30$, $y=?$

$$30^2 + y^2 = 50^2$$
$$900 + y^2 = 2500$$
$$y^2 = 1600$$
$$y = 40$$

$$2 \cdot 30 \cdot 2 + 2 \cdot 40 \cdot \frac{dy}{dt} = 0$$

$$30 \cdot 2 + 40 \cdot \frac{dy}{dt} = 0$$

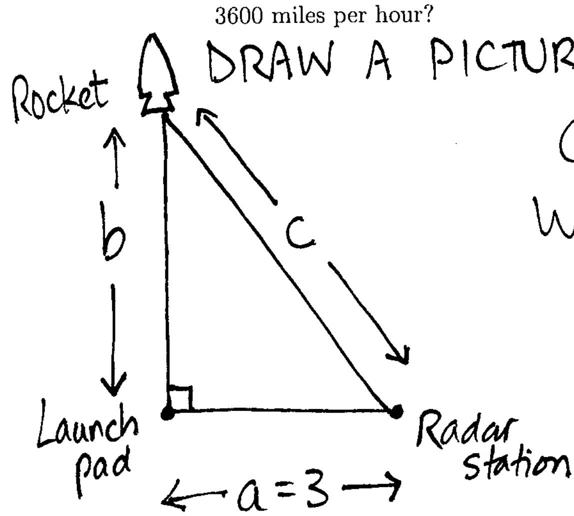
$$40 \cdot \frac{dy}{dt} = -60$$

$$\frac{dy}{dt} = \frac{-60}{40}$$

$$= -\frac{6}{4} = -\frac{3}{2}$$

or -1.5

Question 6. A rocket, rising vertically, is tracked by a radar station that is on the ground 3 miles from the launchpad. How fast is the rocket rising when it is 4 miles high and its distance from the radar station is increasing at a rate of 3600 miles per hour?



DRAW A PICTURE and MAKE UP NAMES!

Given: $a=3$

WANT: Rate at which b is changing

i.e. WANT $\frac{db}{dt}$

At the moment when $b=4$

Given: The distance c is changing at the rate of 3600 mi/hr, i.e. $\frac{dc}{dt} = 3600$

From geometry, we know $a^2+b^2=c^2$

$$9+b^2=c^2$$

$$\frac{d}{dt}(9+b^2) = \frac{d}{dt}(c^2)$$

$$0+2b\cdot\frac{db}{dt} = 2c\cdot\frac{dc}{dt}$$

Note: If $b=4$, then $9+4^2=c^2$

$$9+16=c^2$$

$$25=c^2$$

$$c=5$$

$$2\cdot 4\cdot\frac{db}{dt} = 2\cdot 5\cdot 3600$$

$$4\cdot\frac{db}{dt} = 5\cdot 3600$$

$$\frac{db}{dt} = \frac{5\cdot 3600}{4} = 5\cdot 900 = 4500$$

miles per hour

Tips for remembering linear approximation

(new f) = (old f) + (change in f) and (change in f) = $\frac{df}{dx} \cdot$ (change in x)

$$f(x) \approx f(a) + f'(a)\cdot(x-a)$$

or $f(x_0) + f'(x_0)\cdot(x-x_0)$

Question 7. Find the local linear approximation of the function

$$f(x) = (1+x)^{1/4}$$

at $x_0 = 0$, and use it to estimate $1.04^{1/4}$ and $0.92^{1/4}$.

$$f(x) = (1+x)^{1/4}$$

$$f'(x) = \frac{1}{4}(1+x)^{-3/4} \cdot \underbrace{(1+x)'}_{=1} = \frac{1}{4}(1+x)^{-3/4}$$

or $f(a)$ $\quad f(x_0) = f(0) = (1+0)^{1/4} = 1$

$$f'(x_0) = f'(0) = \frac{1}{4}(1+0)^{-3/4} = \frac{1}{4}(1)^{-3/4} = \frac{1}{4}\cdot 1 = \frac{1}{4}$$

$$f(x) \approx f(0) + f'(0)\cdot(x-0)$$

$$= 1 + \frac{1}{4}\cdot(x-0) = 1 + \frac{1}{4}x$$

$$1.04^{1/4} = (1+0.04)^{1/4} = f(0.04) \approx 1 + \frac{1}{4}(0.04)$$

$$= 1 + 0.01 = \underline{1.01}$$

$$0.92^{1/4} = (1-0.08)^{1/4} = f(-0.08) \approx 1 + \frac{1}{4}(-0.08)$$

$$= 1 - 0.02 = \underline{0.98}$$

Question 8. Use linear approximations to estimate the cube root of 8.06. Start by choosing an appropriate function $f(x)$.

Could choose $f(x) = x^{1/3}$ and $x_0 = a = 8$ because we know $8^{1/3}$ exactly.

$$f(x) = x^{1/3}$$

$$f'(x) = \frac{1}{3}x^{-2/3}$$

$$f(a) = f(8) = 8^{1/3} = (2^3)^{1/3} = 2$$

$$f'(a) = f'(8) = \frac{1}{3}(8)^{-2/3} = \frac{1}{3}(2^3)^{-2/3}$$

$$= \frac{1}{3}\cdot 2^{-2} = \frac{1}{3}\cdot\frac{1}{4} = \frac{1}{12}$$

Linear approximation: $f(x) \approx f(a) + f'(a)\cdot(x-a)$

$$x^{1/3} \approx 2 + \frac{1}{12}\cdot(x-8)$$

So $8.06^{1/3} \approx 2 + \frac{1}{12}\cdot(8.06-8) = 2 + \frac{1}{12}\cdot(0.06)$

$= 2 + 0.005 = 2.005$

Question 9. Evaluate the limit.

$$\lim_{x\to 0} \frac{1-\cos 5x}{x^2}$$

$$\frac{1-\cos(5\cdot 0)}{0^2} = \frac{1-\cos 0}{0} = \frac{1-1}{0} = \frac{0}{0}$$

So we're allowed to use L'Hopital, which gives us

$$\lim_{x\to 0} \frac{-(-\sin 5x \cdot 5)}{2x} = \lim_{x\to 0} \frac{5\sin 5x}{2x}$$

$$\frac{5\sin(5\cdot 0)}{2\cdot 0} = \frac{5\sin 0}{0} = \frac{5\cdot 0}{0} = \frac{0}{0} \text{ again}$$

So we can use L'Hopital again, giving us

$$\lim_{x\to 0} \frac{5\cos 5x \cdot 5}{2} = \frac{5\cos(0)\cdot 5}{2} = \frac{25}{2}$$

Regarding Question 9:

Many students wrote "$\frac{0}{0} = 0$".

This is a very serious mistake!
If you get to know numbers, it should just "feel" wrong.

Remember $\frac{0}{0}$ is shorthand for certain limit problems

and really means $\frac{\text{NEAR zero}}{\text{NEAR zero}}$.

WHAT DOES IT "FEEL" LIKE if a number near zero
is divided by another number near zero?

For example, that could be something like

$\frac{0.000\,003}{0.000\,001}$ which is 3! $\frac{\text{three millionths}}{\text{one millionth}}$

Maybe when you're doing things quickly,
you see $\frac{0}{0}$ and vaguely think
"I'm combining two zeros, so I get zero"
but THAT'S NOT TRUE IF YOU'RE DIVIDING!!

Question 10. Evaluate the limit.

$$\lim_{x\to\infty}(5+\ln x)^{1/x^3}$$

Let $y = (5+\ln x)^{1/x^3}$. We want $\lim\limits_{x\to\infty} y$.

Consider $\ln y = \ln\left((5+\ln x)^{1/x^3}\right)$

$$\ln y = \frac{1}{x^3}\ln(5+\ln x) = \frac{\ln(5+\ln x)}{x^3}$$

What is $\lim\limits_{x\to\infty} \ln y$? $\dfrac{\ln(5+\ln\infty)}{\infty^3} = \dfrac{\ln(5+\infty)}{\infty} = \dfrac{\infty}{\infty}$

So we can use L'Hopital.

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\ln(5+\ln x)}{x^3} = \lim_{x\to\infty}\frac{\frac{1}{5+\ln x}\cdot\frac{1}{x}}{3x^2}$$

$$= \lim_{x\to\infty}\frac{1}{(5+\ln x)\cdot x\cdot 3x^2} = 0 \quad \left(\text{of the form } \frac{1}{\infty}\right)$$

As $x\to\infty$, $\ln y$ approaches 0

so y approaches 1