

MAC2311 Section U10
Suggested problems for Test 3
(Test 3 is Wednesday November 15th, in class)

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1. Find dy/dx .

$$y = e^{7x} \quad \frac{dy}{dx} = e^{7x} \cdot 7 = 7e^{7x}.$$

Chain rule

Other methods: $y = e^u$ and $u = 7x$

$$\frac{dy}{du} = e^u \text{ and } \frac{du}{dx} = 7 \Rightarrow \frac{dy}{dx} = e^u \cdot 7 = e^{7x} \cdot 7$$

2. Find dy/dx .

$$y = e^{-5x^2}$$

$$\frac{dy}{dx} = e^{-5x^2} \cdot (-5x^2)' = e^{-5x^2} \cdot (-10x)$$

or $-10x e^{-5x^2}$

3. Find dy/dx .

$$y = x^3 e^x$$

$$\frac{dy}{dx} = (x^3)' e^x + x^3 (e^x)'$$

$$= 3x^2 e^x + x^3 e^x \quad \text{or} \quad (3x^2 + x^3) e^x$$

Also, we could take logs, and you might want to practice that.

$$\ln y = \ln(x^3 e^x) = \ln(x^3) + \ln(e^x) = 3 \ln x + x$$

$$\text{Then } \frac{d}{dx}(\ln y) = \frac{d}{dx}(3 \ln x + x) \text{ which means } \frac{1}{y} \cdot y' = \frac{3}{x} + 1$$

$$\text{Then } y' = y \cdot \left(\frac{3}{x} + 1\right) = x^3 e^x \cdot \left(\frac{3}{x} + 1\right) = e^x \cdot (3x^2 + x^3)$$

4. Find dy/dx .

$$y = e^{1/x} = e^{x^{-1}}$$

of the form
 e^u
or $e^{g(x)}$

$$\frac{dy}{dx} = e^{x^{-1}} \cdot (x^{-1})'$$

$$= e^{x^{-1}} \cdot (-1x^{-2}) \text{ or } e^{1/x} \cdot \frac{-1}{x^2}$$

5. Find dy/dx .

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\text{or } -\frac{e^{1/x}}{x^2}$$

$$\frac{dy}{dx} = \frac{(e^x - e^{-x})'(e^x + e^{-x}) - (e^x - e^{-x})(e^x + e^{-x})'}{(e^x + e^{-x})^2}$$

$$= \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

Can stop there, but it might be good practice to try to simplify.

$$\frac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})}{(e^x + e^{-x})^2}$$

$$= \frac{4}{(e^x + e^{-x})^2}$$

6. Find dy/dx using any correct method.

$$y = x^{\sin x}$$

$$\ln y = \ln(x^{\sin x}) = (\sin x) \cdot \ln x$$

$$\frac{d}{dx} [\ln y] = \frac{d}{dx} [\sin x \cdot \ln x]$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \cos x \cdot \ln x + \sin x \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = x^{\sin x} \cdot \left(\cos x \cdot \ln x + \frac{\sin x}{x} \right)$$

7. Find dy/dx using any correct method.

$$y = (x^3 - 2x)^{\ln x}$$

$$\ln y = \ln((x^3 - 2x)^{\ln x}) = \ln x \cdot \ln(x^3 - 2x)$$

$$\frac{d}{dx} [\ln y] = \frac{d}{dx} [\ln x \cdot \ln(x^3 - 2x)]$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{x} \cdot \ln(x^3 - 2x) + \ln x \cdot \frac{1}{x^3 - 2x} \cdot (3x^2 - 2)$$

$$\frac{dy}{dx} = y \cdot \left(\frac{\ln(x^3 - 2x)}{x} + \frac{(3x^2 - 2)\ln x}{x^3 - 2x} \right)$$

$$\text{or } (x^3 - 2x)^{\ln x} \cdot \left(\frac{\ln(x^3 - 2x)}{x} + \frac{(3x^2 - 2)\ln x}{x^3 - 2x} \right)$$

This is the INVERSE FUNCTION
of the sine function.

It is NOT $\frac{1}{\sin x}$.

8. Find dy/dx .

$$y = \sin^{-1} x$$

One option: Just memorize it. $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$.

Other method: $y = \sin^{-1} x \Rightarrow \sin y = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
 $\Rightarrow \frac{d}{dx} [\sin y] = \frac{d}{dx} [x] \Rightarrow \cos y \cdot \frac{dy}{dx} = 1$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

9. Find dy/dx .

$$y = \frac{1}{\tan^{-1} x} = (\tan^{-1} x)^{-1}$$

$$\frac{dy}{dx} = -1 (\tan^{-1} x)^{-2} \cdot \frac{d}{dx} (\tan^{-1} x)$$

$$= \boxed{-1 (\tan^{-1} x)^{-2} \cdot \frac{1}{1+x^2}}$$

Got this by just remembering $\frac{d}{dx} [\tan^{-1} x] = \frac{1}{1+x^2}$.

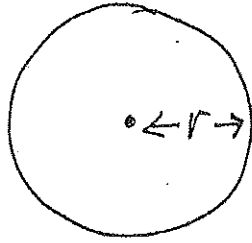
If you don't remember that, you can work it out:

$$y = \tan^{-1} x \Rightarrow \tan y = x \Rightarrow \frac{d}{dx} [\tan y] = \frac{d}{dx} [x]$$

$$\Rightarrow \sec^2 y \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

10. There is a circular oil spill whose radius is increasing at a constant rate of 2 ft/sec. How fast is the area of the spill increasing when the radius of the spill is 60 ft?

DRAW A PICTURE and MAKE UP NAMES.



Let r = radius

A = area

GEOMETRY FACTS WE KNOW:

$$A = \pi r^2$$

GIVEN: r is increasing at a rate of 2 ft/sec

That is, $\frac{dr}{dt} = 2$

WANT: $\frac{dA}{dt}$ (rate at which area is increasing)

at the moment in time when $r = 60$.

$$A = \pi r^2 \Rightarrow \frac{d}{dt} [A] = \frac{d}{dt} [\pi r^2]$$

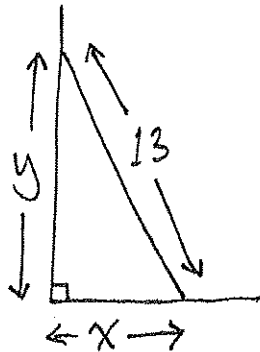
A and r are functions of t

$$\Rightarrow \frac{dA}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt}$$

$$\frac{dA}{dt} = \pi \cdot 2 \cdot 60 \cdot 2 = 240\pi \text{ square feet per second}$$

11. A 13 ft ladder is leaning against a wall. If the top of the ladder slips down the wall at 2 ft/sec, how fast will the bottom be moving away from the wall when the top is 5 ft above the ground?

DRAW A PICTURE and MAKE UP NAMES.



GEOMETRY FACTS WE KNOW:

Right-angled triangle

$$\Rightarrow x^2 + y^2 = 13^2$$

x and y
are functions
of time

GIVEN: Distance y is decreasing at a rate of 2 ft/sec

$$\Rightarrow \frac{dy}{dt} = -2$$

WANT: Rate at which distance x is changing (increasing)

i.e. WANT $\frac{dx}{dt}$, at the moment when distance y is 5.

Note: At the moment when $y=5$, we have $x^2 + y^2 = 13^2$
 $\Rightarrow x^2 + 5^2 = 13^2 \Rightarrow x^2 + 25 = 169 \Rightarrow x^2 = 144 \Rightarrow x = 12$

$$x^2 + y^2 = 13^2 \Rightarrow \frac{d}{dt} [x^2 + y^2] = \frac{d}{dt} [13^2]$$

$$\Rightarrow 2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0 \quad \text{Then plug in } \frac{dy}{dt} = -2, y=5, x=12$$

$$2 \cdot 12 \cdot \frac{dx}{dt} + 2 \cdot 5 \cdot (-2) = 0 \Rightarrow 24 \frac{dx}{dt} - 20 = 0$$

$$\Rightarrow 24 \frac{dx}{dt} = 20 \Rightarrow \frac{dx}{dt} = \frac{20}{24} = \frac{5}{6} \quad x \text{ is increasing at a rate of } \frac{5}{6} \text{ feet/sec.}$$

12. Find the local linear approximation of the function $f(x) = \sqrt{1+x}$ at $x_0 = 0$, and use it to estimate $\sqrt{0.9}$ and $\sqrt{1.1}$.

$$f(x) = (1+x)^{1/2}$$

$$f'(x) = \frac{1}{2}(1+x)^{-1/2} \cdot 1$$

$$f(x_0) = f(0) = (1+0)^{1/2} = 1$$

$$f'(x_0) = f'(0) = \frac{1}{2}(1+0)^{-1/2} = \frac{1}{2}$$

Linear approximation: $f(x) \approx f(x_0) + f'(x_0) \cdot (x - x_0)$

$$= f(0) + f'(0) \cdot (x - 0) = 1 + \frac{1}{2}x$$

$$\sqrt{0.9} = \sqrt{1+(-0.1)} = f(-0.1) \approx 1 + \frac{1}{2}(-0.1) = 1 - 0.05 = 0.95$$

$$\sqrt{1.1} = \sqrt{1+0.1} = f(0.1) \approx 1 + \frac{1}{2}(0.1) = 1 + 0.05 = 1.05$$

13. Find the local linear approximation of the function $f(x) = \frac{1}{1+x}$ at $x_0 = 0$, and use it to estimate $\frac{1}{1.02}$ and $\frac{1}{1.03}$.

$$f(x) = (1+x)^{-1} \quad f'(x) = -1(1+x)^{-2} \cdot 1$$

$$f(x_0) = f(0) = (1+0)^{-1} = 1$$

$$f'(x_0) = f'(0) = -1(1+0)^{-2} = -1$$

Linear approximation: $f(x) \approx f(x_0) + f'(x_0) \cdot (x - x_0)$

$$= f(0) + f'(0) \cdot (x - 0) = 1 + (-1)x = 1 - x$$

$$\frac{1}{1.02} = \frac{1}{1+0.02} = f(0.02) \approx 1 - 0.02 = 0.98$$

$$\frac{1}{1.03} = \frac{1}{1+0.03} = f(0.03) \approx 1 - 0.03 = 0.97$$

14. Evaluate the limit.

$$\lim_{x \rightarrow 0} \frac{x^2}{\sin x} \quad \text{This has the form } \frac{0}{0}$$

$$\text{L'Hopital} \rightarrow \lim_{x \rightarrow 0} \frac{2x}{\cos x} = \frac{2 \cdot 0}{\cos 0} = \frac{0}{1} = 0$$

15. Evaluate the limit.

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{\sin x} \quad \text{This has the form } \frac{0}{0}$$

$$\text{L'Hopital: } \lim_{x \rightarrow 0} \frac{e^x}{\cos x} = \frac{e^0}{\cos 0} = \frac{1}{1} = 1$$

16. Evaluate the limit.

$$\lim_{x \rightarrow 0} \frac{\sin 2x}{\sin 5x} \quad \text{This has the form } \frac{0}{0}$$

$$\text{L'Hopital: } \lim_{x \rightarrow 0} \frac{\cos(2x) \cdot 2}{\cos(5x) \cdot 5} = \frac{\cos(0) \cdot 2}{\cos(0) \cdot 5} = \frac{1 \cdot 2}{1 \cdot 5} = \frac{2}{5}$$

17. Evaluate the limit.

$$\lim_{x \rightarrow 0} (1-3x)^{1/x} \quad \text{Let } L = (1-3x)^{1/x}$$

$$\text{Then } \ln L = \ln((1-3x)^{1/x}) = \frac{1}{x} \ln(1-3x) = \frac{\ln(1-3x)}{x}$$

When $x \rightarrow 0$, $\frac{\ln(1-3x)}{x}$ has the form $\frac{\ln 1}{0} = \frac{0}{0}$ so we can use L'Hopital

$\ln L$ approaches $\lim_{x \rightarrow 0} \frac{\frac{1}{1-3x} \cdot (-3)}{1} = \lim_{x \rightarrow 0} \frac{-3}{1-3x} = \frac{-3}{1-0} = -3$. That is, $\ln L \rightarrow -3$
so $L \rightarrow e^{-3}$.

18. Evaluate the limit.

$$\lim_{x \rightarrow \infty} (\ln x)^{1/x} \quad \text{Let } L = (\ln x)^{1/x}$$

$$\text{Then } \ln L = \ln((\ln x)^{1/x}) = \frac{1}{x} \ln(\ln x) = \frac{\ln(\ln x)}{x}$$

When $x \rightarrow \infty$, $\ln L = \frac{\ln(\ln x)}{x}$ has the form $\frac{\infty}{\infty}$ so we can use L'Hopital

$\ln L$ approaches $\lim_{x \rightarrow \infty} \frac{\frac{1}{\ln x} \cdot \frac{1}{x}}{1} = \lim_{x \rightarrow \infty} \frac{1}{x \ln x} = \frac{1}{\infty} = 0$. That is, $\ln L \rightarrow 0$
so $L \rightarrow e^0 = 1$.

19. Evaluate the limit.

$$\lim_{x \rightarrow \infty} x^{1/\ln x} \quad \text{Let } L = x^{1/\ln x}$$

$$\text{Then } \ln L = \ln(x^{1/\ln x}) = \frac{1}{\ln x} \ln x = 1!$$

Almost a trick question, but the good kind of trick question that's actually easier to answer than appears at first glance!

$\ln L$ just is 1, so L is $e^1 = e$

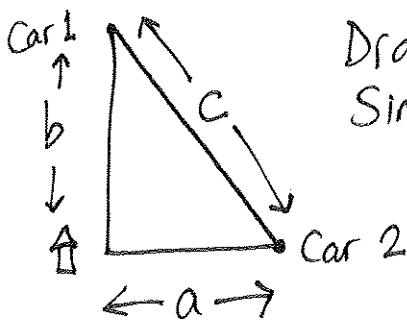
So certainly L approaches e .

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Answer is e .

20. Two cars leave my house at noon. One drives straight north at a uniform speed of 40 miles per hour, and the other drives straight east at a uniform speed of 30 miles per hour. At exactly 1 pm, find the rate at which the diagonal distance between the cars is increasing.

At 1 pm, the first car has traveled north for 1 hour and the second car has traveled east for 1 hour. So the first car is 40 miles north of my house and the second car is 30 miles east of my house.



Draw a picture and make up variable names. Since the cars are moving, the distances are changing.

Given: Car #1 moves at rate of 40 mi/hr $\Rightarrow \frac{db}{dt} = 40$

Given: Car #2 moves at rate of 30 mi/hr $\Rightarrow \frac{da}{dt} = 30$

WANT: the rate at which distance c is changing at 1 pm

i.e. we WANT $\frac{dc}{dt}$ at 1 pm. Remember: At 1 pm, $b=40$ and $a=30$

Geometry tells us $a^2 + b^2 = c^2$. Then $\frac{d}{dt}(a^2 + b^2) = \frac{d}{dt}(c^2)$

$$\Rightarrow 2a \cdot \frac{da}{dt} + 2b \cdot \frac{db}{dt} = 2c \cdot \frac{dc}{dt} \Rightarrow a \cdot \frac{da}{dt} + b \cdot \frac{db}{dt} = c \cdot \frac{dc}{dt}$$

At 1 pm, we have $b=40$ and $a=30$, so $a^2 + b^2 = c^2 \Rightarrow 30^2 + 40^2 = c^2$
 $\Rightarrow 900 + 1600 = c^2 \Rightarrow 2500 = c^2 \Rightarrow c = 50$.

Then $\underbrace{30}_a \cdot \underbrace{30}_{\frac{da}{dt}} + \underbrace{40}_b \cdot \underbrace{40}_{\frac{db}{dt}} = \underbrace{50}_c \cdot \frac{dc}{dt} \Rightarrow \dots$ (ran out of room on this page)

$\dots \Rightarrow \frac{dc}{dt} = 50 \text{ mi/h}$