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MAC 2311 Section U20 Test 2
Friday October 27th
Total possible score: 20 points (2 points per page)

Question 1. Find dy/dx.

$$y = 7x^4 + 3x^2 - 163$$

$$\frac{dy}{dx} = 7 \cdot 4x^3 + 3 \cdot 2x + 0$$

$$= 28x^3 + 6x$$

Question 2a. Write down the **definition** of the derivative.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Question 2b. Use the **definition** of the derivative to find the derivative of $f(x) = \sqrt{x}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{\left(\sqrt{x+h}\right)^2 - \left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h \to 0} \frac{x + h - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Question 3. Find $f'(x)$, and simplify.

$$f(x) = \frac{2x^{3/2}+1}{2x^{1/2}+1}$$

$$f'(x) = \frac{(2x^{3/2}+1)'(2x^{1/2}+1) - (2x^{3/2}+1)(2x^{1/2}+1)'}{(2x^{1/2}+1)^2}$$

$$= \frac{\left(2\cdot\frac{3}{2}x^{1/2}+0\right)\left(2x^{1/2}+1\right) - \left(2x^{3/2}+1\right)\left(2\cdot\frac{1}{2}x^{-1/2}+0\right)}{(2x^{1/2}+1)^2}$$

$$= \frac{3x^{1/2}(2x^{1/2}+1) - (2x^{3/2}+1)x^{-1/2}}{(2x^{1/2}+1)^2}$$

$$= \frac{6x+3x^{1/2}-2x-x^{-1/2}}{(2x^{1/2}+1)^2}$$

$$= \frac{4x+3x^{1/2}-x^{-1/2}}{(2x^{1/2}+1)^2}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Question 4. Find the **second** derivative $f''(x)$.

$$f(x) = \frac{3x+5}{4x+7}$$

$$f'(x) = \frac{(3x+5)'(4x+7) - (3x+5)(4x+7)'}{(4x+7)^2}$$

$$= \frac{3\cdot(4x+7) - (3x+5)\cdot 4}{(4x+7)^2} = \frac{(12x+21) - (12x+20)}{(4x+7)^2}$$

$$= \frac{12x + 21 - 12x - 20}{(4x+7)^2} = \frac{1}{(4x+7)^2} \quad \text{OR} \quad (4x+7)^{-2}$$

So $f'(x) = (4x+7)^{-2}$ is of the form u^{-2} or $(g(x))^{-2}$ and we can use the chain rule

$$f''(x) = -2(4x+7)^{-3}\cdot(4x+7)'$$

$$= -2(4x+7)^{-3}\cdot 4$$

or $-8(4x+7)^{-3}$

or $\dfrac{-8}{(4x+7)^3}$

Question 5. Find $f'(x)$ and <u>simplify</u>.

$$f(x) = \frac{7 - \cos x}{7 + \sin x}$$

$$f'(x) = \frac{(7-\cos x)'(7+\sin x) - (7-\cos x)(7+\sin x)'}{(7+\sin x)^2}$$

$$= \frac{(0-(-\sin x))(7+\sin x) - (7-\cos x)(0+\cos x)}{(7+\sin x)^2}$$

$$= \frac{\sin x(7+\sin x) - (7-\cos x)\cos x}{(7+\sin x)^2}$$

$$= \frac{(7\sin x + \sin^2 x) - (7\cos x - \cos^2 x)}{(7+\sin x)^2}$$

$$= \frac{7\sin x + \sin^2 x - 7\cos x + \cos^2 x}{(7+\sin x)^2}$$

$$= \frac{7\sin x - 7\cos x + 1}{(7+\sin x)^2}$$

Question 6. Find $f'(x)$ using any correct method.

$$f(x) = \left(\frac{x^2-1}{x^2+1}\right)^{37}$$

METHOD 1: $f = u^{37}$ and $u = \dfrac{x^2-1}{x^2+1}$

so $f'(x) = \dfrac{df}{dx} = \dfrac{df}{du}\cdot\dfrac{du}{dx} = 37u^{36}\cdot\dfrac{du}{dx}$

$$= 37\left(\frac{x^2-1}{x^2+1}\right)^{36}\cdot\left(\frac{x^2-1}{x^2+1}\right)'$$

$$= 37\left(\frac{x^2-1}{x^2+1}\right)^{36}\cdot\frac{2x\cdot(x^2+1)-(x^2-1)\cdot 2x}{(x^2+1)^2}$$

$$= 37\left(\frac{x^2-1}{x^2+1}\right)^{36}\cdot\frac{2x^3+2x-2x^3+2x}{(x^2+1)^2} = 37\left(\frac{x^2-1}{x^2+1}\right)^{36}\cdot\frac{4x}{(x^2+1)^2}$$

METHOD 2: $\ln f(x) = \ln\left(\left(\dfrac{x^2-1}{x^2+1}\right)^{37}\right) = 37\ln\left(\dfrac{x^2-1}{x^2+1}\right)$

$= 37\left(\ln(x^2-1)-\ln(x^2+1)\right)$. Then take $\frac{d}{dx}$ of each side

$$\Rightarrow \frac{1}{f(x)}\cdot f'(x) = 37\left(\frac{1}{x^2-1}\cdot 2x - \frac{1}{x^2+1}\cdot 2x\right)$$

$$f'(x) = 37f(x)\cdot\left(\frac{2x}{x^2-1} - \frac{2x}{x^2+1}\right)$$

which can be rewritten in various ways

Question 7. Find $f'(x)$.

$$f(x) = \sin^3\left(\frac{x}{x+1}\right)$$

or equivalently $$f(x) = \left(\sin\left(\frac{x}{x+1}\right)\right)^3$$

$f = (\text{something})^3$

$f = u^3$ and $u = \sin\left(\frac{x}{x+1}\right)$

$\downarrow$

$u = \sin v$ and $v = \frac{x}{x+1}$

$$\frac{df}{du} = 3u^2 \qquad \frac{du}{dv} = \cos v \qquad \frac{dv}{dx} = \frac{(x)'(x+1)-(x)(x+1)'}{(x+1)^2}$$

$$= \frac{1(x+1)-x\cdot 1}{(x+1)^2} = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2}$$

$$f'(x) = \frac{df}{dx} = \frac{df}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}$$

$$= 3u^2 \cdot \cos v \cdot \frac{1}{(x+1)^2}$$

$$= 3\sin^2\left(\frac{x}{x+1}\right)\cdot\cos\left(\frac{x}{x+1}\right)\cdot\frac{1}{(x+1)^2}$$

Question 8. Find dy/dx using any correct method.

$$x^3 + y^3 = 3xy^2$$

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}\left(3x \cdot y^2\right)$$

Product rule

$$\frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(y^3\right) = (3x)' \cdot y^2 + 3x \cdot \left(y^2\right)'$$

where prime means $\frac{d}{dx}$

chain rule

$$3x^2 + 3y^2 \cdot y' = 3 \cdot y^2 + 3x \cdot 2y \cdot y'$$

y is a FUNCTION of x

chain rule

$$3x^2 + 3y^2 \cdot y' = 3y^2 + 6xy \cdot y'$$

$$-3x^2 \qquad -6xy \cdot y' \qquad\qquad -3x^2 \qquad -6xy \cdot y'$$

$$3y^2 \cdot y' - 6xy \cdot y' = 3y^2 - 3x^2$$

$$\left(3y^2 - 6xy\right) y' = 3y^2 - 3x^2$$

$$y' = \frac{3y^2 - 3x^2}{3y^2 - 6xy} = \frac{3\left(y^2 - x^2\right)}{3\left(y^2 - 2xy\right)} = \frac{y^2 - x^2}{y^2 - 2xy}$$

Question 9. Find dy/dx using any correct method.

$$y = \ln\left(\frac{x}{x^2+1}\right)$$

METHOD 1: $y = \ln u$ and $u = \dfrac{x}{x^2+1}$

$$\frac{dy}{du} = \frac{1}{u} \qquad \frac{du}{dx} = \frac{(x)'(x^2+1)-(x)(x^2+1)'}{(x^2+1)^2}$$

$$= \frac{1(x^2+1)-x\cdot 2x}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{u}\cdot\frac{1-x^2}{(x^2+1)^2}$$

$$= \frac{x^2+1}{x}\cdot\frac{1-x^2}{(x^2+1)^2} = \frac{1-x^2}{x(x^2+1)}$$

METHOD 2: $y = \ln\left(\dfrac{x}{x^2+1}\right) = \ln x - \ln(x^2+1)$

$$\frac{dy}{dx} = \frac{1}{x} - \frac{1}{x^2+1}\cdot(x^2+1)' = \frac{1}{x} - \frac{2x}{x^2+1}$$

which can also be written $\dfrac{x^2+1}{x(x^2+1)} - \dfrac{2x^2}{x(x^2+1)}$

or $\dfrac{1-x^2}{x(x^2+1)}$

Question 10. Find dy/dx using any correct method.

$$y = \ln\sqrt{\frac{x+1}{x-1}}$$

METHOD 1: $y = \ln u$ and $u = \sqrt{\frac{x+1}{x-1}} \Rightarrow u = \sqrt{v} = v^{1/2}$ and $v = \frac{x+1}{x-1}$

$$\frac{dy}{du} = \frac{1}{u}$$

$$\frac{du}{dv} = \frac{1}{2}v^{-1/2} \quad \text{or} \quad \frac{1}{2}\cdot\frac{1}{\sqrt{v}}$$

$$\frac{dv}{dx} = \frac{(x+1)'(x-1)-(x+1)(x-1)'}{(x-1)^2} = \frac{1\cdot(x-1)-(x+1)\cdot 1}{(x-1)^2} = \frac{-2}{(x-1)^2}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx} = \frac{1}{u}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{v}}\cdot\frac{-2}{(x-1)^2}$$

$$= \sqrt{\frac{x-1}{x+1}}\cdot\frac{1}{2}\cdot\sqrt{\frac{x-1}{x+1}}\cdot\frac{-2}{(x-1)^2} = -1\cdot\frac{x-1}{x+1}\cdot\frac{1}{(x-1)^2}$$

$$= \frac{-1}{(x+1)(x-1)}$$

METHOD 2: $y = \ln\left(\left(\frac{x+1}{x-1}\right)^{1/2}\right)$

$$= \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) = \frac{1}{2}\Big(\ln(x+1) - \ln(x-1)\Big).$$ Then take $\frac{d}{dx}$ of each side

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{x+1}\cdot 1 - \frac{1}{x-1}\cdot 1\right) = \frac{1}{2}\left(\frac{1}{x+1} - \frac{1}{x-1}\right)$$

$$= \frac{1}{2}\left(\frac{x-1}{(x+1)(x-1)} - \frac{x+1}{(x+1)(x-1)}\right) = \frac{1}{2}\left(\frac{-2}{(x+1)(x-1)}\right)$$

$$= \frac{-1}{(x+1)(x-1)}$$