MAC2311 Section U20
Suggested problems for Test 3
(Test 3 is Friday November 17th, in class)

Idris Mercer

November 7, 2017

1. Find dy/dx.

$$y = e^{7x}$$

$\frac{dy}{dx} = e^{7x} \cdot 7 = 7e^{7x}$.

chain rule

Other methods: $y = e^u$ and $u = 7x$

$\frac{dy}{du} = e^u$ and $\frac{du}{dx} = 7 \Rightarrow \frac{dy}{dx} = e^u \cdot 7 = e^{7x} \cdot 7$

2. Find dy/dx.

$$y = e^{-5x^2}$$

$\frac{dy}{dx} = e^{-5x^2} \cdot (-5x^2)' = e^{-5x^2} \cdot (-10x)$

or $-10x e^{-5x^2}$

3. Find dy/dx.

$$y = x^3 e^x$$

$\frac{dy}{dx} = (x^3)' e^x + x^3 (e^x)'$

$= 3x^2 e^x + x^3 e^x$ or $(3x^2 + x^3)e^x$

Also, we could take logs, and you might want to practice that.

$\ln y = \ln(x^3 e^x) = \ln(x^3) + \ln(e^x) = 3\ln x + x$

Then $\frac{d}{dx}(\ln y) = \frac{d}{dx}(3\ln x + x)$ which means $\frac{1}{y} \cdot y' = \frac{3}{x} + 1$

Then $y' = y \cdot \left(\frac{3}{x} + 1\right) = x^3 e^x \cdot \left(\frac{3}{x} + 1\right) = e^x \cdot (3x^2 + x^3)$

4. Find dy/dx.

$$y = e^{1/x} = e^{x^{-1}}$$

of the form e^u or $e^{g(x)}$

$$\frac{dy}{dx} = e^{x^{-1}} \cdot (x^{-1})'$$

$$= e^{x^{-1}} \cdot (-1x^{-2}) \text{ or } e^{1/x} \cdot \frac{-1}{x^2} \text{ or } \frac{-e^{1/x}}{x^2}$$

5. Find dy/dx.

$$y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$\frac{dy}{dx} = \frac{(e^x - e^{-x})'(e^x + e^{-x}) - (e^x - e^{-x})(e^x + e^{-x})'}{(e^x + e^{-x})^2}$$

$$= \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

Can stop there, but it might be good practice to try to simplify.

$$\frac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})}{(e^x + e^{-x})^2}$$

$$= \frac{4}{(e^x + e^{-x})^2}$$

6. Find dy/dx using any correct method.

$$y = x^{\sin x}$$

$$\ln y = \ln\left(x^{\sin x}\right) = (\sin x)\cdot \ln x$$

$$\frac{d}{dx}\left[\ln y\right] = \frac{d}{dx}\left[\sin x \cdot \ln x\right]$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \cos x \cdot \ln x + \sin x \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = x^{\sin x}\cdot\left(\cos x \cdot \ln x + \frac{\sin x}{x}\right)$$

7. Find dy/dx using any correct method.

$$y = (x^3 - 2x)^{\ln x}$$

$$\ln y = \ln\left((x^3-2x)^{\ln x}\right) = \ln x \cdot \ln(x^3-2x)$$

$$\frac{d}{dx}\left[\ln y\right] = \frac{d}{dx}\left[\ln x \cdot \ln(x^3-2x)\right]$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{x}\cdot \ln(x^3-2x) + \ln x \cdot \frac{1}{x^3-2x}\cdot(3x^2-2)$$

$$\frac{dy}{dx} = y\cdot\left(\frac{\ln(x^3-2x)}{x} + \frac{(3x^2-2)\ln x}{x^3-2x}\right)$$

$$\text{or } (x^3-2x)^{\ln x}\cdot\left(\frac{\ln(x^3-2x)}{x} + \frac{(3x^2-2)\ln x}{x^3-2x}\right)$$

This is the INVERSE FUNCTION of the sine function. It is NOT $\frac{1}{\sin x}$.

8. Find dy/dx.

$$y = \sin^{-1} x$$

One option: Just memorize it. $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$.

Other method: $y = \sin^{-1} x \Rightarrow \sin y = x$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$$\Rightarrow \frac{d}{dx}[\sin y] = \frac{d}{dx}[x] \Rightarrow \cos y \cdot \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

9. Find dy/dx.

$$y = \frac{1}{\tan^{-1} x} = (\tan^{-1} x)^{-1}$$

$$\frac{dy}{dx} = -1(\tan^{-1} x)^{-2} \cdot \frac{d}{dx}(\tan^{-1} x)$$

$$= \boxed{-1(\tan^{-1} x)^{-2} \cdot \frac{1}{1+x^2}}$$

Got this by just <u>remembering</u> $\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2}$.

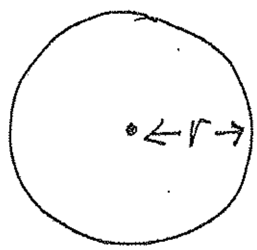
If you don't remember that, you can work it out:

$$y = \tan^{-1} x \Rightarrow \tan y = x \Rightarrow \frac{d}{dx}[\tan y] = \frac{d}{dx}[x]$$

$$\Rightarrow \sec^2 y \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

10. There is a circular oil spill whose radius is increasing at a constant rate of 2 ft/sec. How fast is the area of the spill increasing when the radius of the spill is 60 ft?

DRAW A PICTURE and MAKE UP NAMES.



Let r = radius

A = area

GEOMETRY FACTS WE KNOW:

$$A = \pi r^2$$

GIVEN: r is increasing at a rate of 2 ft/sec

That is, $\frac{dr}{dt} = 2$

WANT: $\frac{dA}{dt}$ (rate at which area is increasing)

at the moment in time when $r = 60$.

$$A = \pi r^2 \Rightarrow \frac{d}{dt}[A] = \frac{d}{dt}[\pi r^2]$$

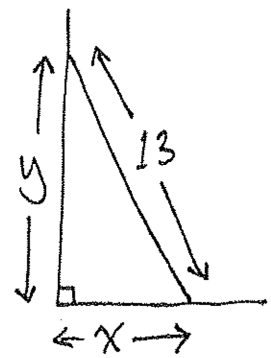
A and r are functions of t

$$\Rightarrow \frac{dA}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt}$$

$$\frac{dA}{dt} = \pi \cdot 2 \cdot 60 \cdot 2 = 240\pi$$ square feet per second

11. A 13 ft ladder is leaning against a wall. If the top of the ladder slips down the wall at 2 ft/sec, how fast will the bottom be moving away from the wall when the top is 5 ft above the ground?

DRAW A PICTURE and MAKE UP NAMES.



GEOMETRY FACTS WE KNOW:

Right-angled triangle

$\Rightarrow x^2 + y^2 = 13^2$

x and y are functions of time

GIVEN: Distance y is decreasing at a rate of 2 ft/sec

$$\Rightarrow \frac{dy}{dt} = -2$$

WANT: Rate at which distance x is changing (increasing)

i.e. WANT $\frac{dx}{dt}$, at the moment when distance y is 5.

Note: At the moment when $y=5$, we have $x^2+y^2=13^2$

$$\Rightarrow x^2+5^2=13^2 \Rightarrow x^2+25=169 \Rightarrow x^2=144 \Rightarrow x=12$$

$$x^2+y^2=13^2 \Rightarrow \frac{d}{dt}\left[x^2+y^2\right]=\frac{d}{dt}\left[13^2\right]$$

$$\Rightarrow 2x\cdot\frac{dx}{dt}+2y\cdot\frac{dy}{dt}=0$$ Then plug in $\frac{dy}{dt}=-2$, $y=5$, $x=12$

$$2\cdot 12\cdot\frac{dx}{dt}+2\cdot 5\cdot(-2)=0 \Rightarrow 24\frac{dx}{dt}-20=0$$

$$\Rightarrow 24\frac{dx}{dt}=20 \Rightarrow \frac{dx}{dt}=\frac{20}{24}=\frac{5}{6}.$$

x is increasing at a rate of $\frac{5}{6}$ feet/sec.

12. Find the local linear approximation of the function $f(x) = \sqrt{1+x}$ at $x_0 = 0$, and use it to estimate $\sqrt{0.9}$ and $\sqrt{1.1}$.

$f(x) = (1+x)^{1/2}$

$f'(x) = \frac{1}{2}(1+x)^{-1/2} \cdot 1$

$f(x_0) = f(0) = (1+0)^{1/2} = 1$

$f'(x_0) = f'(0) = \frac{1}{2}(1+0)^{-1/2} = \frac{1}{2}$

Linear approximation: $f(x) \approx f(x_0) + f'(x_0) \cdot (x - x_0)$

$= f(0) + f'(0) \cdot (x - 0) = 1 + \frac{1}{2}x$

$\sqrt{0.9} = \sqrt{1+(-0.1)} = f(-0.1) \approx 1 + \frac{1}{2}(-0.1) = 1 - 0.05 = 0.95$

$\sqrt{1.1} = \sqrt{1+0.1} = f(0.1) \approx 1 + \frac{1}{2}(0.1) = 1 + 0.05 = 1.05$

13. Find the local linear approximation of the function $f(x) = \frac{1}{1+x}$ at $x_0 = 0$, and use it to estimate $\frac{1}{1.02}$ and $\frac{1}{1.03}$.

$f(x) = (1+x)^{-1}$ $\quad f'(x) = -1(1+x)^{-2} \cdot 1$

$f(x_0) = f(0) = (1+0)^{-1} = 1$

$f'(x_0) = f'(0) = -1(1+0)^{-2} = -1$

Linear approximation: $f(x) \approx f(x_0) + f'(x_0) \cdot (x - x_0)$

$= f(0) + f'(0) \cdot (x - 0) = 1 + (-1)x = 1 - x$

$\frac{1}{1.02} = \frac{1}{1+0.02} = f(0.02) \approx 1 - 0.02 = 0.98$

$\frac{1}{1.03} = \frac{1}{1+0.03} = f(0.03) \approx 1 - 0.03 = 0.97$

14. Evaluate the limit.

$$\lim_{x\to 0} \frac{x^2}{\sin x}$$ This has the form $\frac{0}{0}$

L'Hopital $\to$ $\lim\limits_{x\to 0} \dfrac{2x}{\cos x} = \dfrac{2\cdot 0}{\cos 0} = \dfrac{0}{1} = 0$

15. Evaluate the limit.

$$\lim_{x\to 0} \frac{e^x - 1}{\sin x}$$ This has the form $\frac{0}{0}$

L'Hopital: $\lim\limits_{x\to 0} \dfrac{e^x}{\cos x} = \dfrac{e^0}{\cos 0} = \dfrac{1}{1} = 1$

16. Evaluate the limit.

$$\lim_{x\to 0} \frac{\sin 2x}{\sin 5x}$$ This has the form $\frac{0}{0}$

L'Hopital: $\lim\limits_{x\to 0} \dfrac{\cos(2x)\cdot 2}{\cos(5x)\cdot 5} = \dfrac{\cos(0)\cdot 2}{\cos(0)\cdot 5} = \dfrac{1\cdot 2}{1\cdot 5}$

$= \dfrac{2}{5}$

17. Evaluate the limit.

$\lim_{x\to 0}(1-3x)^{1/x}$ Let $L = (1-3x)^{1/x}$

Then $\ln L = \ln\left((1-3x)^{1/x}\right) = \frac{1}{x}\ln(1-3x) = \frac{\ln(1-3x)}{x}$

When $x \to 0$, $\frac{\ln(1-3x)}{x}$ has the form $\frac{\ln 1}{0} = \frac{0}{0}$ so we can use L'Hopital

$\ln L$ approaches $\lim_{x\to 0} \frac{\frac{1}{1-3x}\cdot(-3)}{1} = \lim_{x\to 0}\frac{-3}{1-3x} = \frac{-3}{1-0} = -3.$ That is, $\ln L \to -3$

so $L \to e^{-3}$

18. Evaluate the limit.

$\lim_{x\to\infty}(\ln x)^{1/x}$ Let $L = (\ln x)^{1/x}$

Then $\ln L = \ln\left((\ln x)^{1/x}\right) = \frac{1}{x}\ln(\ln x) = \frac{\ln(\ln x)}{x}$

When $x\to\infty$, $\ln L = \frac{\ln(\ln x)}{x}$ has the form $\frac{\infty}{\infty}$ so we can use L'Hopital

$\ln L$ approaches $\lim_{x\to\infty}\frac{\frac{1}{\ln x}\cdot\frac{1}{x}}{1} = \lim_{x\to\infty}\frac{1}{x\ln x} = \frac{1}{\infty} = 0.$ That is, $\ln L \to 0$

So $L \to e^0 = 1.$

19. Evaluate the limit.

$\lim_{x\to\infty} x^{1/\ln x}$ Let $L = x^{1/\ln x}$

Then $\ln L = \ln\left(x^{1/\ln x}\right) = \frac{1}{\ln x}\ln x = 1$!

Almost a trick question, but the good kind of trick question that's actually easier to answer than appears at first glance!

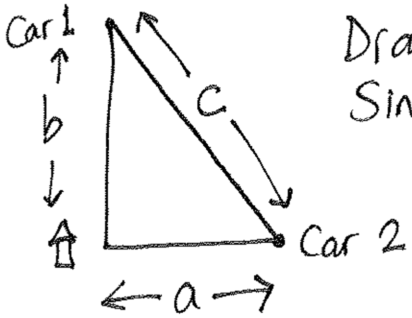
$\ln L$ just is 1, so L is $e^1 = e$

So certainly L approaches e.

Answer is e.

20. Two cars leave my house at noon. One drives straight north at a uniform speed of 40 miles per hour, and the other drives straight east at a uniform speed of 30 miles per hour. At exactly 1 pm, find the rate at which the diagonal distance between the cars is increasing.

At 1 pm, the first car has traveled north for 1 hour and the second car has traveled east for 1 hour.
So the first car is 40 miles north of my house and the second car is 30 miles east of my house.



Draw a picture and make up variable names.
Since the cars are moving, the distances are changing.

Given: Car #1 moves at rate of 40 mi/hr $\Rightarrow \frac{db}{dt} = 40$

Given: Car #2 moves at rate of 30 mi/hr $\Rightarrow \frac{da}{dt} = 30$

WANT: the rate at which distance c is changing at 1 pm
i.e. we WANT $\frac{dc}{dt}$ at 1 pm. Remember: At 1 pm, $b=40$ and $a=30$

Geometry tells us $a^2+b^2=c^2$. Then $\frac{d}{dt}(a^2+b^2) = \frac{d}{dt}(c^2)$

$\Rightarrow 2a \cdot \frac{da}{dt} + 2b \cdot \frac{db}{dt} = 2c \cdot \frac{dc}{dt} \Rightarrow a \cdot \frac{da}{dt} + b \cdot \frac{db}{dt} = c \cdot \frac{dc}{dt}$

At 1 pm, we have $b=40$ and $a=30$, so $a^2+b^2=c^2 \Rightarrow 30^2+40^2=c^2$
$\Rightarrow 900 + 1600 = c^2 \Rightarrow 2500 = c^2 \Rightarrow c = 50$.

Then $\underbrace{30}_{a} \cdot \underbrace{30}_{\frac{da}{dt}} + \underbrace{40}_{b} \cdot \underbrace{40}_{\frac{db}{dt}} = \underbrace{50}_{c} \cdot \frac{dc}{dt} \Rightarrow \ldots$ (ran out of room on this page)

$\ldots \Rightarrow \frac{dc}{dt} = 50$ mi/h