MAC2312
Suggested problems on Chapter 5 material (integration)

Idris Mercer

January 12, 2018

1. Evaluate the sum.

$$\sum_{k=1}^{3} k^3$$

$$1^3 + 2^3 + 3^3 = 1 + 8 + 27 = 36$$

2. Evaluate the sum.

$$\sum_{j=2}^{6} (3j-1)$$

$$(3\cdot 2-1) + (3\cdot 3-1) + (3\cdot 4-1) + (3\cdot 5-1) + (3\cdot 6-1)$$

$$= 5 + 8 + 11 + 14 + 17 = 55$$

OR $\displaystyle\sum_{j=2}^{6} 3j + \sum_{j=2}^{6}(-1) = 3\sum_{j=2}^{6} j + 5\cdot(-1)$

$$= 3(2+3+4+5+6) - 5 = 3\cdot 20 - 5 = 55$$

3. Evaluate the sum.

$$\sum_{i=-4}^{1}(i^2 - i) \quad = \sum_{i=-4}^{1}\left(i\cdot(i-1)\right)$$

$$= \left(-4\cdot(-5)\right) + \left(-3\cdot(-4)\right) + \left(-2\cdot(-3)\right) + \left(-1\cdot(-2)\right) + \left(0\cdot(-1)\right) + \left(1\cdot 0\right)$$

$$= \quad 20 \quad + \quad 12 \quad + \quad 6 \quad + 2 + 0 + 0$$

$$= \quad 40$$

or of course $\left(16-(-4)\right)+\left(9-(-3)\right)+\left(4-(-2)\right)+\left(1-(-1)\right)+\left(0-0\right)+\left(1-1\right)$

4. Evaluate the sum.

$$\sum_{n=0}^{5} 1 \quad = 6$$

$$\underset{n=0}{1} + \underset{n=1}{1} + \underset{n=2}{1} + \underset{n=3}{1} + \underset{n=4}{1} + \underset{n=5}{1}$$

5. Evaluate the sum.

$$\sum_{k=0}^{4}(-2)^k$$

$$(-2)^0+(-2)^1+(-2)^2+(-2)^3+(-2)^4$$

$$= 1 + (-2) + 4 + (-8) + 16$$

$$= (1+4+16)-(2+8) = 21-10 = 11$$

6. Evaluate the sum.

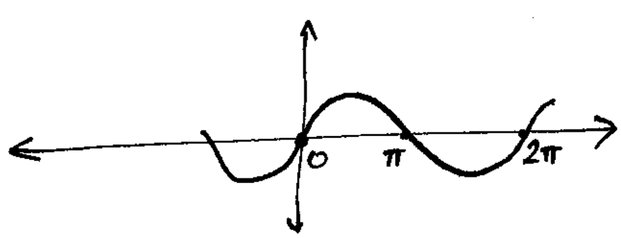
$$\sum_{n=1}^{6}\sin n\pi$$

$$\sin\pi + \sin 2\pi + \sin 3\pi + \sin 4\pi + \sin 5\pi + \sin 6\pi$$

$$= 0 + 0 + 0 + 0 + 0 + 0$$

$$= 0$$

$y = \sin x$:



7. Write the expression in sigma notation but do not evaluate.

$$1 + 2 + 3 + \cdots + 10$$

$$\sum_{k=1}^{10} k$$

8. Write the expression in sigma notation but do not evaluate.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$$

$$\sum_{k=1}^{5} \frac{(-1)^{k+1}}{k} \quad \text{or} \quad \sum_{k=1}^{5} \frac{(-1)^{k-1}}{k}$$

$$\text{or} \quad \sum_{k=0}^{4} \frac{(-1)^{k}}{k+1}$$

Facts: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

9. Use formulas given in the textbook (Theorem 5.4.2) to evaluate the sum.

$$\sum_{k=1}^{100} k = \frac{100\cdot 101}{2} = 50\cdot 101$$

$$= 5050$$

10. Use formulas given in the textbook (Theorem 5.4.2) to evaluate the sum.

$$\sum_{k=1}^{30} k(k-2)(k+2)$$

$$\sum_{k=1}^{30} k(k^2-4) = \sum_{k=1}^{30}(k^3-4k)$$

$$= \sum_{k=1}^{30} k^3 - 4\sum_{k=1}^{30} k$$

$$= \left(\frac{30\cdot 31}{2}\right)^2 - 4\cdot\frac{30\cdot 31}{2}$$

$$= (15\cdot 31)^2 - 4\cdot 15\cdot 31$$

$$= 15\cdot 31\cdot(15\cdot 31 - 4) = 465\cdot 461$$

which would be annoying to work out by hand.

It's 214365

11. Express the sum in closed form.

$$\sum_{k=1}^{n} \frac{3k}{n}$$

$$\sum_{k=1}^{n} \left(\frac{3}{n} \cdot k \right)$$

$$= \frac{3}{n} \sum_{k=1}^{n} k$$

$$= \frac{3}{n} \cdot \frac{n(n+1)}{2} = \frac{3(n+1)}{2}$$

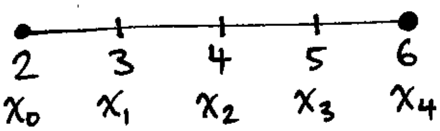
12. Let f be the function $f(x) = 3x+1$ on the interval $[2, 6]$. Divide the interval into $n = 4$ subintervals of equal length and then compute

$$\sum_{k=1}^{4} f(x_k^*)\Delta x$$

if
(a) x_k^* is the left endpoint of each subinterval,
(b) x_k^* is the midpoint of each subinterval,
(c) x_k^* is the right endpoint of each subinterval.
Illustrate each part with a graph of f that includes the rectangles whose areas are represented in the sum.

$$\Delta x = \frac{6-2}{4} = \frac{4}{4} = 1$$

$x_0 = 2$
$x_1 = 3$
$x_2 = 4$
$x_3 = 5$
$x_4 = 6$



(a) $x_1^* = 2 \quad x_2^* = 3 \quad x_3^* = 4 \quad x_4^* = 5$

(b) $x_1^* = 2.5 \quad x_2^* = 3.5 \quad x_3^* = 4.5 \quad x_4^* = 5.5$

(c) $x_1^* = 3 \quad x_2^* = 4 \quad x_3^* = 5 \quad x_4^* = 6$

x:	2	2.5	3	3.5	4	4.5	5	5.5	6
$f(x)=3x+1$:	7	8.5	10	11.5	13	14.5	16	17.5	19

(a) $f(2)\cdot 1 + f(3)\cdot 1 + f(4)\cdot 1 + f(5)\cdot 1 = 7+10+13+16$
$= 46$

(b) $f(2.5)\cdot 1 + f(3.5)\cdot 1 + f(4.5)\cdot 1 + f(5.5)\cdot 1 = 8.5+11.5+14.5+17.5$
$= 52$

(c) $f(3)\cdot 1 + f(4)\cdot 1 + f(5)\cdot 1 + f(6)\cdot 1 = 10+13+16+19$
$= 58$

13. Consider the sum

$$\sum_{k=1}^{4}\left((k+1)^3 - k^3\right).$$

Notice that this is

$$(2^3 - 1^3) + (3^3 - 2^3) + (4^3 - 3^3) + (5^3 - 4^3)$$

which is the same as

$$(5^3 - 4^3) + (4^3 - 3^3) + (3^3 - 2^3) + (2^3 - 1^3)$$

which simplifies to $5^3 - 1^3$, which is $125 - 1 = 124$. (This type of sum is called a **telescoping** sum.) Use a similar idea to evaluate

$$\sum_{k=5}^{17}(3^k - 3^{k-1}).$$

$$(3^5 - 3^4) + (3^6 - 3^5) + (3^7 - 3^6) + \dots$$

$$\dots + (3^{15} - 3^{14}) + (3^{16} - 3^{15}) + (3^{17} - 3^{16})$$

$$= 3^{17} - 3^4$$

which could also be written

$$3^4 \cdot (3^{13} - 1)$$

But 3^{13} or 3^{17} take too long to evaluate by hand

So we can just leave our answer in that form.

14. Consider the sum

$$S = \sum_{k=0}^{n} ar^k.$$

Show that $S - rS = a - ar^{n+1}$, which implies that

$$S = \frac{a - ar^{n+1}}{1 - r} \qquad \text{if } r \neq 1.$$

$$S = a + ar + ar^2 + \ldots + ar^{n-1} + ar^n$$

$$rS = ar + ar^2 + ar^3 + \ldots + ar^n + ar^{n+1}$$

$$S - rS = a - ar^{n+1}$$

$$S \cdot (1 - r) = a - ar^{n+1}$$

$$S = \frac{a - ar^{n+1}}{1 - r}$$

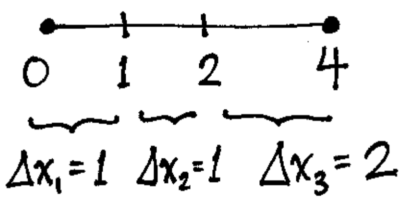
15. Find the value of

$$\sum_{k=1}^{n} f(x_k^*)\,\Delta x_k$$

if

$$f(x) = x + 1,$$
$$a = 0, \qquad b = 4, \qquad n = 3,$$
$$\Delta x_1 = 1, \qquad \Delta x_2 = 1, \qquad \Delta x_3 = 2,$$
$$x_1^* = \frac{1}{3}, \qquad x_2^* = \frac{3}{2}, \qquad x_3^* = 3.$$

It may help to draw a picture of the interval.



$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + f(x_3^*)\Delta x_3$$
$$= f\left(\tfrac{1}{3}\right)\cdot 1 + f\left(\tfrac{3}{2}\right)\cdot 1 + f(3)\cdot 2$$
$$= \left(\tfrac{1}{3} + 1\right) + \left(\tfrac{3}{2} + 1\right) + (3+1)\cdot 2$$
$$= \frac{4}{3} + \frac{5}{2} + 8$$
$$= \frac{8}{6} + \frac{15}{6} + \frac{48}{6} = \frac{71}{6}$$

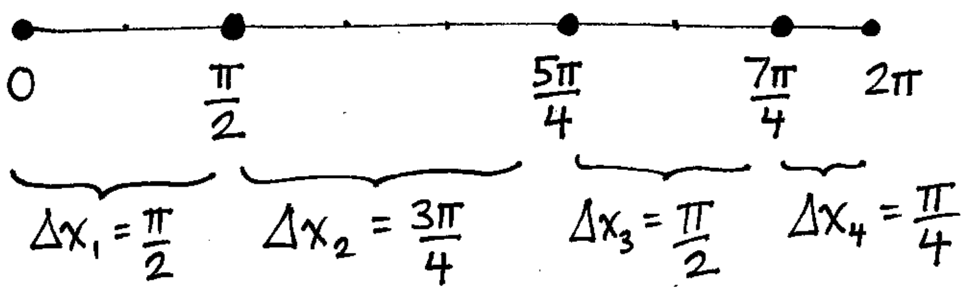
16. Find the value of

$$\sum_{k=1}^{n} f(x_k^*)\,\Delta x_k$$

if

$$f(x) = \cos x,$$
$$a = 0, \qquad b = 2\pi, \qquad n = 4,$$
$$\Delta x_1 = \pi/2, \qquad \Delta x_2 = 3\pi/4, \qquad \Delta x_3 = \pi/2, \qquad \Delta x_4 = \pi/4$$
$$x_1^* = \pi/4, \qquad x_2^* = \pi, \qquad x_3^* = 3\pi/2, \qquad x_4^* = 7\pi/4.$$

It may help to draw a picture of the interval.



$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + f(x_3^*)\Delta x_3 + f(x_4^*)\Delta x_4$$

$$= f\left(\frac{\pi}{4}\right)\cdot\frac{\pi}{2} + f(\pi)\cdot\frac{3\pi}{4} + f\left(\frac{3\pi}{2}\right)\cdot\frac{\pi}{2} + f\left(\frac{7\pi}{4}\right)\cdot\frac{\pi}{4}$$

$$= \cos\left(\frac{\pi}{4}\right)\cdot\frac{\pi}{2} + \cos(\pi)\cdot\frac{3\pi}{4} + \cos\left(\frac{3\pi}{2}\right)\cdot\frac{\pi}{2} + \cos\left(\frac{7\pi}{4}\right)\cdot\frac{\pi}{4}$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{\pi}{2} + (-1)\cdot\frac{3\pi}{4} + 0\cdot\frac{\pi}{2} + \frac{\sqrt{2}}{2}\cdot\frac{\pi}{4}$$

$$= \frac{\pi\sqrt{2}}{4} - \frac{3\pi}{4} + \frac{\pi\sqrt{2}}{8} = \frac{2\pi\sqrt{2}}{8} - \frac{6\pi}{8} + \frac{\pi\sqrt{2}}{8}$$

$$= \frac{3\pi\sqrt{2} - 6\pi}{8} \quad \text{or} \quad \frac{3\pi(\sqrt{2} - 2)}{8}$$

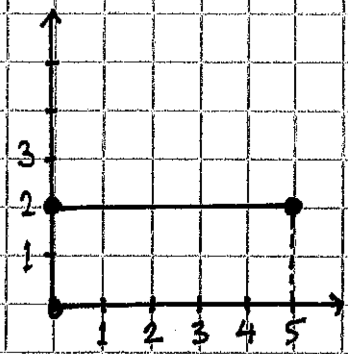
17. Sketch the region whose signed area is represented by the definite integral, and evaluate the integral using your knowledge of geometry.

(a) $\displaystyle\int_0^5 2\,dx$

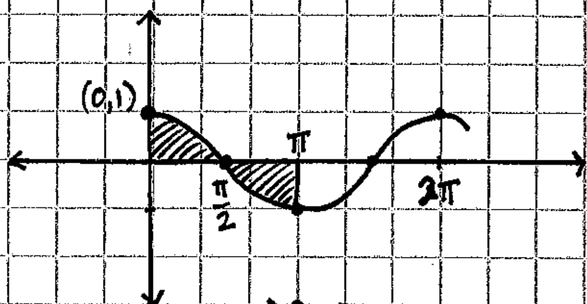
(b) $\displaystyle\int_0^{\pi} \cos x\,dx$

(c) $\displaystyle\int_{-1}^{2} |2x-3|\,dx$

(d) $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\,dx$

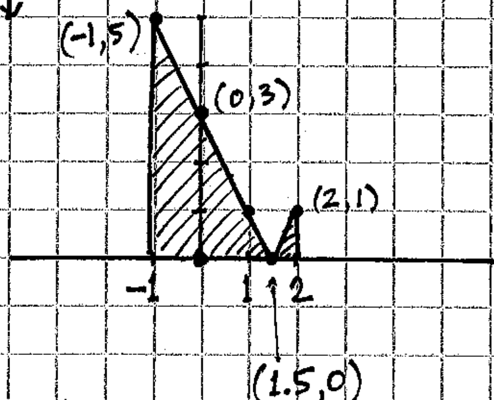


$$\int_0^5 2\,dx = 5 \cdot 2 = 10$$



$$\int_0^{\pi} \cos x\,dx = 0$$

Net signed area is 0
because of symmetry of graph

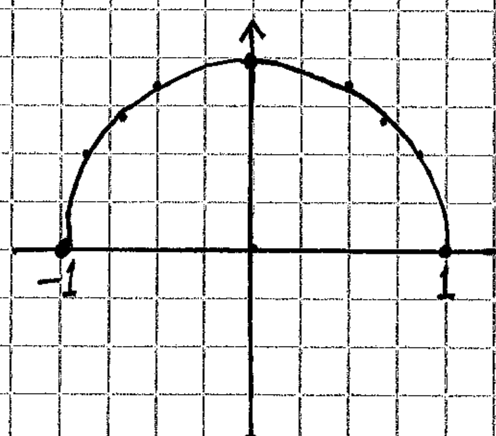


$$\int_{-1}^{2} |2x-3|\,dx = \text{total area of two triangles}$$

First triangle: Height 5, base $\frac{5}{2}$

Second triangle: Height 1, base $\frac{1}{2}$

$$\text{Area} = \frac{1}{2}\cdot\frac{5}{2}\cdot 5 + \frac{1}{2}\cdot\frac{1}{2}\cdot 1 = \frac{25}{4} + \frac{1}{4} = \frac{26}{4}$$



$$y = \sqrt{1-x^2} \Rightarrow y^2 = 1 - x^2$$

$$x^2 + y^2 = 1$$

Top half of unit circle

$$\text{Area} = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi \cdot 1^2$$

$$= \frac{\pi}{2}$$

18. If f is the function defined by

$$f(x) = \begin{cases} |x-2|, & \text{if } x \geq 0, \\ x+2, & \text{if } x < 0, \end{cases}$$

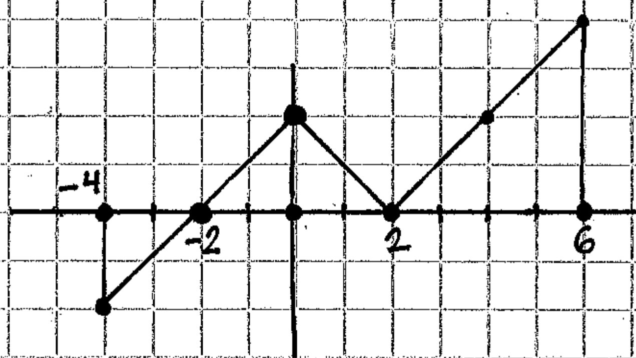
then evaluate each of the integrals.

(a) $\displaystyle\int_{-2}^{0} f(x)\,dx$

(b) $\displaystyle\int_{-2}^{2} f(x)\,dx$

(c) $\displaystyle\int_{0}^{6} f(x)\,dx$

(d) $\displaystyle\int_{-4}^{6} f(x)\,dx$



$$\int_{-2}^{0} f(x)\,dx = 2$$

$$\int_{-2}^{2} f(x)\,dx = 4$$

$$\int_{0}^{6} f(x)\,dx = \frac{4}{2} + \frac{16}{2} = 2+8 = 10$$

$$\int_{-4}^{6} f(x)\,dx = -2 + 2 + 2 + 8 = 10$$

19. Find $\int_{-1}^{2} (f(x) + 2g(x))\, dx$ if

$$\int_{-1}^{2} f(x)\, dx = 5 \qquad \text{and} \qquad \int_{-1}^{2} g(x)\, dx = -3.$$

$$\int_{-1}^{2} \left(f(x) + 2g(x) \right) dx$$

$$= \int_{-1}^{2} f(x)\, dx + \int_{-1}^{2} 2g(x)\, dx$$

$$= \int_{-1}^{2} f(x)\, dx + 2\int_{-1}^{2} g(x)\, dx$$

$$= 5 + 2\cdot(-3)$$

$$= 5 - 6 = -1$$

20. Evaluate the integral by completing the square and using your knowledge of geometry.

$$\int_0^{10} \sqrt{10x - x^2}\, dx$$

$$10x - x^2 = -1(x^2 - 10x)$$
$$= -1(x^2 - 10x + 25 - 25)$$
$$= -1\left((x-5)^2 - 25 \right) = 25 - (x-5)^2$$

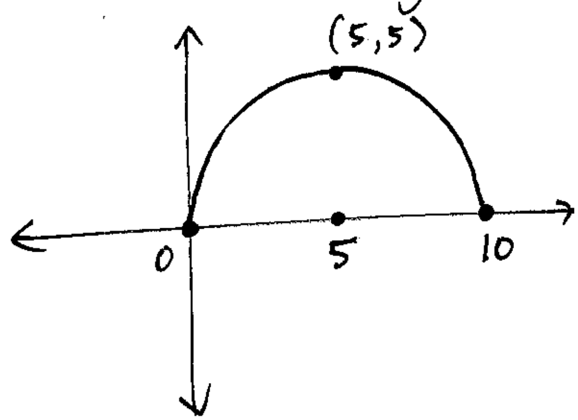
so integral $= \displaystyle\int_0^{10} \sqrt{25 - (x-5)^2}\, dx$

Now note that $y = \sqrt{25 - (x-5)^2}$

implies $y^2 = 25 - (x-5)^2 \Rightarrow (x-5)^2 + y^2 = 25$

so graph is top half of a circle of radius 5

centered at $(x, y) = (5, 0)$



$$\text{Area} = \frac{1}{2}\pi r^2$$
$$= \frac{1}{2}\pi \cdot 5^2 = \frac{25\pi}{2}$$