21. Find the area under the curve $y = x^3$ over the interval $[2, 3]$.

$$\int_2^3 x^3\,dx = \left[\frac{x^4}{4}\right]_2^3 = \frac{1}{4}\left[x^4\right]_2^3$$

$$= \frac{1}{4}\left(3^4 - 2^4\right) = \frac{1}{4}(81 - 16) = \frac{65}{4}$$

22. Find the area under the curve $y = e^{2x}$ over the interval $[0, \ln 2]$.

$$\int_0^{\ln 2} e^{2x}\,dx = \left[\frac{1}{2}e^{2x}\right]_0^{\ln 2} = \frac{1}{2}\left[e^{2x}\right]_0^{\ln 2}$$

$$= \frac{1}{2}\left(e^{2\ln 2} - e^0\right) = \frac{1}{2}\left(\left(e^{\ln 2}\right)^2 - 1\right) = \frac{1}{2}\left(2^2 - 1\right)$$

$$= 3/2$$

23. Find the area under the curve $y = \frac{1}{x}$ over the interval $[1, 5]$.

$$\int_1^5 \frac{1}{x}\,dx = \left[\ln|x|\right]_1^5 = \ln|5| - \ln|1|$$

$$= \ln 5 - \ln 1 = \ln 5 - 0$$

$$= \ln 5$$

24. Evaluate the integral.

$$\int_{-2}^{1} (x^2 - 6x + 12)\, dx$$

$$\left[\frac{x^3}{3} - 6\frac{x^2}{2} + 12x \right]_{-2}^{1}$$

$$= \frac{1}{3}\left[x^3\right]_{-2}^{1} - 3\left[x^2\right]_{-2}^{1} + 12\left[x\right]_{-2}^{1}$$

$$= \frac{1}{3}\left(1-(-8)\right) - 3\left(1-4\right) + 12\left(1-(-2)\right)$$

$$= \frac{9}{3} - 3\cdot(-3) + 12\cdot 3 = 3 + 9 + 36 = 48$$

25. Evaluate the integral.

$$\int_{0}^{\pi/4} \sec^2\theta \, d\theta$$

$$\left[\tan\theta\right]_{\theta=0}^{\theta=\pi/4} = \tan\frac{\pi}{4} - \tan 0$$

$$= 1 - 0$$

$$= 1$$

26. Evaluate the integral.

$$\int_{-1}^{1} |2x-1| \, dx$$

Note: $|w| = \begin{cases} w & \text{if } w \geq 0 \\ -w & \text{if } w < 0 \end{cases}$ "changes" at $w=0$

So $|2x-1| = \begin{cases} 2x-1 & \text{if } 2x-1 \geq 0 \\ -(2x-1) & \text{if } 2x-1 < 0 \end{cases}$ "changes" at $2x-1=0$

$= 1-2x$ $\qquad 2x=1$

so we should split integral at $x = \frac{1}{2}$ $\qquad x = 1/2$

$$\int_{-1}^{1/2} |2x-1| \, dx + \int_{1/2}^{1} |2x-1| \, dx$$

$$= \int_{-1}^{1/2} (1-2x) \, dx + \int_{1/2}^{1} (2x-1) \, dx$$

$$= \left[x - x^2\right]_{-1}^{1/2} + \left[x^2 - x\right]_{1/2}^{1}$$

$$= \left[x\right]_{-1}^{1/2} - \left[x^2\right]_{-1}^{1/2} + \left[x^2\right]_{1/2}^{1} - \left[x\right]_{1/2}^{1}$$

$$= \left[x\right]_{-1}^{1/2} + \left[x^2\right]_{1/2}^{-1} + \left[x^2\right]_{1/2}^{1} + \left[x\right]_{1}^{1/2}$$

$$= \frac{1}{2} - (-1) + 1 - \frac{1}{4} + 1 - \frac{1}{4} + \frac{1}{2} - 1 = \ldots = \frac{5}{2}$$

27. A particle moves along an s-axis. Use the given information to find the position function of the particle.

(a) $v(t) = 3t^2 - 2t$, $s(0) = 1$

(b) $a(t) = 3\sin 3t$, $v(0) = 3$, $s(0) = 3$.

(a) $s(t) = \int v(t)\,dt = \int (3t^2 - 2t)\,dt$

$= t^3 - t^2 + C$. Plugging in $t=0$, we find $C = s(0)$.

$s(t) = t^3 - t^2 + 1$.

(b) $v(t) = \int a(t)\,dt = \int 3\sin 3t\,dt$

$= -\cos 3t + C$. Plugging in $t=0$, we get $v(0) = -\cos 0 + C$

$3 = -1 + C$

$\Rightarrow C = 4$

$v(t) = 4 - \cos 3t$

$s(t) = \int v(t)\,dt = \int (4 - \cos 3t)\,dt$

$= 4t - \frac{1}{3}\sin 3t + C$. Plugging in $t=0$, we get $s(0) = 0 - \frac{1}{3}\sin 0 + C$

$3 = 0 - 0 + C$

$\Rightarrow C = 3$

$s(t) = 4t - \frac{1}{3}\sin 3t + 3$

28. A particle moves with a velocity of $v(t)$ m/s along an s-axis. Find the displacement and the distance traveled by the particle during the given time interval.

(a) $v(t) = \sin t,\ 0 \le t \le \pi/2$

(b) $v(t) = \cos t,\ \pi/2 \le t \le 2\pi$

I have deliberately skipped some of the problems on rectilinear motion.
They play some role in the course but not a large role.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\pi/6$	$1/2$	$\sqrt{3}/2$	$1/\sqrt{3}$
$\pi/4$	$1/\sqrt{2}$	$1/\sqrt{2}$	1
$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$\pi/2$	1	0	undefined

32. Find the average value of the function over the given interval.

$$f(x) = \sin x, \qquad [0, \pi]$$

$$\frac{1}{\pi - 0}\int_0^{\pi} \sin x \, dx = \frac{1}{\pi}\Big[-\cos x\Big]_0^{\pi}$$

$$= \frac{1}{\pi}\Big[\cos x\Big]_{\pi}^{0} = \frac{1}{\pi}\left(\cos 0 - \cos\pi\right) = \frac{1}{\pi}\left(1-(-1)\right)$$

$$= \frac{2}{\pi}$$

33. Find the average value of the function over the given interval.

$$f(x) = \frac{1}{1+x^2}, \qquad [1, \sqrt{3}]$$

$$\frac{1}{\sqrt{3}-1}\int_1^{\sqrt{3}} \frac{1}{1+x^2}\,dx = \frac{1}{\sqrt{3}-1}\Big[\arctan x\Big]_1^{\sqrt{3}}$$

or $\tan^{-1} x$

$$= \frac{1}{\sqrt{3}-1}\left(\arctan\sqrt{3} - \arctan 1\right)$$

$$= \frac{1}{\sqrt{3}-1}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{1}{\sqrt{3}-1}\left(\frac{4\pi}{12} - \frac{3\pi}{12}\right)$$

$$= \frac{1}{\sqrt{3}-1}\cdot\frac{\pi}{12} \quad \text{or} \quad \frac{\pi}{12(\sqrt{3}-1)}$$

Could multiply by $\frac{\sqrt{3}+1}{\sqrt{3}+1}$ if we want to rationalize denominator

We get $\frac{\pi(\sqrt{3}+1)}{24}$

34a. Suppose that the velocity function of a particle moving along a coordinate line is $v(t) = 3t^3 + 2$. Find the average velocity of the particle over the time interval $1 \le t \le 4$ by integrating.

$$\frac{1}{4-1}\int_1^4 v(t)\,dt = \frac{1}{3}\int_1^4 (3t^3+2)\,dt$$

$$= \frac{1}{3}\left[3\frac{t^4}{4} + 2t\right]_1^4 = \frac{1}{3}\cdot\frac{3}{4}\left[t^4\right]_1^4 + \frac{1}{3}\cdot 2\left[t\right]_1^4$$

$$= \frac{1}{4}(4^4-1) + \frac{2}{3}(4-1) = \frac{255}{4} + 2 = \frac{263}{4}$$

34b. Suppose that the position function of a particle moving along a coordinate line is $s(t) = 6t^2 + t$. Find the average velocity of the particle over the time interval $1 \le t \le 4$ algebraically.

$$\frac{s(4)-s(1)}{4-1} = \frac{(6\cdot 4^2+4)-(6\cdot 1^2+1)}{3}$$

$$= \frac{6\cdot 16 + 4 - 6 - 1}{3} = \frac{96+4-6-1}{3}$$

$$= \frac{93}{3} = 31$$

35. Evaluate the definite integral.

$$\int_0^1 (2x+1)^3 \, dx$$

Try substitution. $u = 2x+1$ $\quad \frac{du}{dx} = 2$

$$du = 2dx$$

$$\frac{1}{2} du = dx$$

$x=0 \Rightarrow u = 2\cdot 0 + 1 = 1$

$x=1 \Rightarrow u = 2\cdot 1 + 1 = 3$

$$\int_{x=0}^{x=1} (2x+1)^3 \, dx = \int_{u=1}^{u=3} u^3 \cdot \frac{1}{2} du$$

$$= \frac{1}{2}\int_1^3 u^3 \, du = \frac{1}{2}\left[\frac{u^4}{4}\right]_{u=1}^{u=3}$$

$$= \frac{1}{8}\left[u^4\right]_{u=1}^{u=3} = \frac{1}{8}\left(3^4 - 1^4\right)$$

$$= \frac{1}{8}(81-1) = \frac{1}{8}(80) = 10$$

36. Evaluate the definite integral.

$$\int_0^{\pi/6} 2\cos 3x \, dx$$

Try $u = 3x$ $\quad x = 0 \Rightarrow u = 3\cdot 0 = 0$

$du = 3dx$ $\quad x = \frac{\pi}{6} \Rightarrow u = 3\cdot\frac{\pi}{6} = \frac{\pi}{2}$

$\frac{1}{3}du = dx$

$$\int_{x=0}^{x=\pi/6} 2\cos 3x \, dx = \int_{u=0}^{u=\pi/2} 2\cos u \cdot \frac{1}{3} du$$

$$= \frac{2}{3}\int_0^{\pi/2} \cos u \, du = \frac{2}{3}\Big[\sin u\Big]_{u=0}^{u=\pi/2}$$

$$= \frac{2}{3}\left(\sin\frac{\pi}{2} - \sin 0\right) = \frac{2}{3}(1-0) = \frac{2}{3}$$

37. Evaluate the definite integral by expressing it in terms of u and evaluating the resulting integral using your knowledge of geometry.

$$\int_{-5/3}^{5/3} \sqrt{25 - 9x^2}\, dx, \qquad u = 3x$$

Note: $u^2 = (3x)^2$
$= 9x^2$

$x = -\frac{5}{3} \Rightarrow u = -5$

$x = \frac{5}{3} \Rightarrow u = 5$

$du = 3dx$

$\frac{1}{3} du = dx$

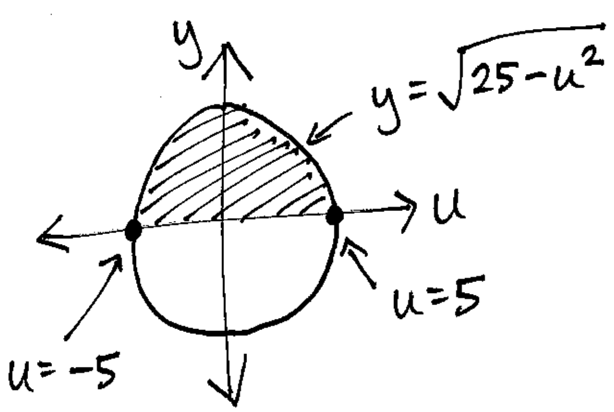
$$\int_{x=-5/3}^{x=5/3} \sqrt{25 - 9x^2}\, dx = \int_{u=-5}^{u=5} \sqrt{25-u^2} \cdot \frac{1}{3}\, du$$

$$= \frac{1}{3} \int_{-5}^{5} \sqrt{25-u^2}\, du.$$

Now consider the graph of $y = \sqrt{25-u^2}$ $u = -5 \ldots 5$

$y^2 = 25 - u^2$

$u^2 + y^2 = 25$ SEMICIRCLE (top half. $y>0$)



$\int_{-5}^{5} \sqrt{25-u^2}\, du$ = shaded area

$= \frac{1}{2} \pi \cdot 5^2 = \frac{25\pi}{2}$

Final answer is $\frac{1}{3}$ times that.

$\frac{25\pi}{6}$

38. Find the area under the curve $y = 9/(x+2)^2$ over the interval $[-1, 1]$.

$$\int_{-1}^{1} \frac{9}{(x+2)^2}\,dx = \int_{-1}^{1} 9(x+2)^{-2}\,dx$$

Try $u = x+2$ $\qquad x=-1 \Rightarrow u = -1+2 = 1$

$du = 1dx = dx \qquad x = 1 \Rightarrow u = 1+2 = 3$

$$\int_{x=-1}^{x=1} 9(x+2)^{-2}\,dx = \int_{u=1}^{u=3} 9u^{-2}\,du$$

$$= \left[9\frac{u^{-1}}{-1} \right]_{u=1}^{u=3} = \left[-\frac{9}{u} \right]_{u=1}^{u=3} = \left[\frac{9}{u} \right]_{u=3}^{u=1}$$

$$= \frac{9}{1} - \frac{9}{3} = 9 - 3 = 6$$

39. Find the average value of $f(x) = x/(5x^2+1)^2$ over the interval $[0, 2]$.

$$\frac{1}{2-0}\int_0^2 \frac{x}{(5x^2+1)^2}\,dx$$

Try substitution

$u = 5x^2 + 1$

$du = 10x\,dx$

$\frac{1}{10}du = x\,dx$

$x = 0 \Rightarrow u = 5\cdot 0^2 + 1 = 1$

$x = 2 \Rightarrow u = 5\cdot 2^2 + 1 = 21$

$$\frac{1}{2}\int_{x=0}^{x=2} (5x^2+1)^{-2}\,x\,dx$$

$$= \frac{1}{2}\int_{u=1}^{u=21} u^{-2}\cdot\frac{1}{10}\,du = \frac{1}{20}\int_{u=1}^{u=21} u^{-2}\,du$$

$$= \frac{1}{20}\left[\frac{u^{-1}}{-1}\right]_{u=1}^{u=21} = \frac{1}{20}\left[-\frac{1}{u}\right]_{u=1}^{u=21}$$

$$= \frac{1}{20}\left[\frac{1}{u}\right]_{u=21}^{u=1} = \frac{1}{20}\left(1 - \frac{1}{21}\right) = \frac{1}{20}\cdot\frac{20}{21}$$

$$= \frac{1}{21}$$

40. Find the derivative.

$$\frac{d}{dx}\int_1^{x^3} \frac{1}{t}\,dt$$

Let $G = \int_1^{x^3} \frac{1}{t}\,dt$. We want $\frac{dG}{dx}$.

$G = \int_1^{u} \frac{1}{t}\,dt$ and $u = x^3$

$\frac{du}{dx} = 3x^2$

$\frac{dG}{du} = \frac{1}{u}$ by Fundamental Theorem

Then by Chain rule, $\frac{dG}{dx} = \frac{dG}{du} \cdot \frac{du}{dx}$

$$= \frac{1}{u} \cdot 3x^2$$

$$= \frac{1}{x^3} \cdot 3x^2 = \frac{3}{x}.$$

Note: It is also possible to evaluate the integral defining G.

$G = \int_1^{x^3} \frac{1}{t}\,dt = \Big[\ln|t|\Big]_{t=1}^{t=x^3} = \ln|x^3| - \ln|1|$

$= \ln|x^3|$. So $\frac{dG}{dx} = \frac{1}{x^3} \cdot 3x^2$ by chain rule.

41a. Find the derivative.

$$\frac{d}{dx}\int_x^\pi \cos(t^3)\, dt$$

Let $G = \int_x^\pi \cos(t^3)\,dt$. We want $\frac{dG}{dx}$.

$$G = -\int_\pi^x \cos(t^3)\,dt$$

$$\frac{dG}{dx} = -\cos(x^3) \quad \text{by Fundamental Theorem.}$$

(Note that this time, we cannot evaluate the integral, but that's OK, because roughly paraphrased, we're taking the derivative of the integral of $\cos(x^3)$.)

41b. Find the derivative.

$$\frac{d}{dx}\int_{\tan x}^{3} \frac{t^2}{1+t^2}\,dt$$

Let $G = \displaystyle\int_{\tan x}^{3} \frac{t^2}{1+t^2}\,dt$. We want $\dfrac{dG}{dx}$.

$$G = -\int_{3}^{\tan x} \frac{t^2}{1+t^2}\,dt$$

$$G = -\int_{3}^{u} \frac{t^2}{1+t^2}\,dt \quad \text{and} \quad u = \tan x$$

$$\frac{dG}{du} = -\frac{u^2}{1+u^2} \qquad \frac{du}{dx} = \sec^2 x$$

$$\frac{dG}{dx} = \frac{dG}{du}\cdot\frac{du}{dx} = -\frac{u^2}{1+u^2}\cdot\sec^2 x$$

$$= -\frac{\tan^2 x}{1+\tan^2 x}\cdot\sec^2 x \quad \text{which can be simplified}$$

$$= \frac{-\tan^2 x}{\sec^2 x}\cdot\sec^2 x = -\tan^2 x.$$

Note that it is possible to integrate $\dfrac{t^2}{1+t^2}$, but we don't need to.