MAC2312
Suggested problems on Chapter 6 material (applications of integration)

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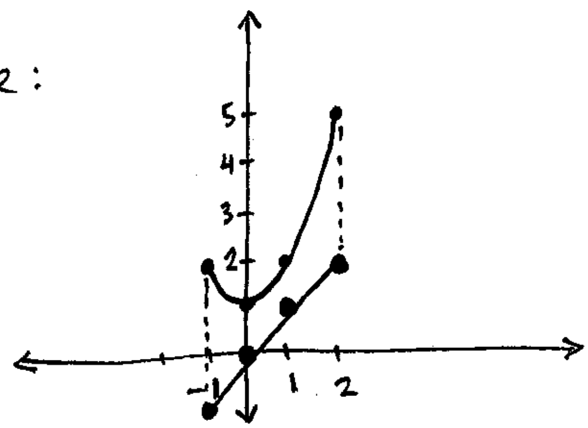
1. Find the area bounded by $y = x$ and $y = x^2 + 1$ between $x = -1$ and $x = 2$.

May help to draw a picture:

$y = x^2 + 1$ is top curve

$y = x$ is bottom curve



$$\text{Area} = \int_{-1}^{2} \underbrace{(\underbrace{x^2+1}_{\text{top}} - \underbrace{x}_{\text{bottom}})}_{\text{height}} \underbrace{dx}_{\text{width}} = \left[\frac{x^3}{3} + x - \frac{x^2}{2} \right]_{-1}^{2}$$

$$= \frac{1}{3}\left[x^3\right]_{-1}^{2} + \left[x\right]_{-1}^{2} - \frac{1}{2}\left[x^2\right]_{-1}^{2}$$

$$= \frac{1}{3}\left(8-(-1)\right) + \left(2-(-1)\right) - \frac{1}{2}(4-1)$$

$$= \frac{9}{3} + 3 - \frac{3}{2} = 3 + 3 - \frac{3}{2} = \frac{12}{2} - \frac{3}{2} = \frac{9}{2}$$

2. Find the area bounded by $y = \sqrt{x}$ and $y = -\frac{1}{4}x$ between $x = 0$ and $x = 4$.

If $0 < x < 4$ then $\sqrt{x}$ has positive outputs $\rightarrow$ top

$-\frac{1}{4}x$ has negative outputs $\rightarrow$ bottom

$$\text{Area} = \int_0^4 \left(\sqrt{x} - \left(-\tfrac{1}{4}x\right)\right) dx$$

$$= \int_0^4 \left(x^{1/2} + \frac{x}{4}\right) dx = \left[\frac{x^{3/2}}{3/2} + \frac{x^2}{8}\right]_0^4$$

$$= \frac{2}{3} \cdot 4^{3/2} + \frac{4^2}{8} = \frac{2}{3} \cdot 2^3 + 2 = \frac{16}{3} + \frac{6}{3}$$

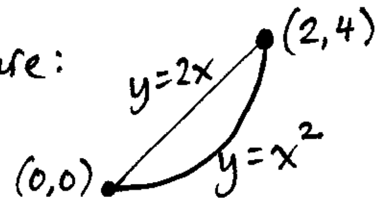
$$= \frac{22}{3}$$

3. Find the area bounded by $y = x^2$ and $y = 2x$. Find the area by integrating with respect to x, **and** by integrating with respect to y, and check that your two answers are equal.

Intersections? $x^2 = 2x \Rightarrow x^2 - 2x = 0$

$x(x-2) = 0 \Rightarrow x=0,\ x=2$

Rough picture:



Integrating with respect to x: Top is $y = 2x$, bottom is $y = x^2$

$$\text{Area} = \int_{x=0}^{x=2} (2x - x^2)\,dx = \left[x^2 - \frac{x^3}{3}\right]_0^2 = 2^2 - \frac{2^3}{3}$$

$$= 4 - \frac{8}{3} = \frac{12}{3} - \frac{8}{3} = \frac{4}{3}$$

Integrating with respect to y: $x = \frac{1}{2}y$ is left curve

$x = \sqrt{y}$ is right curve

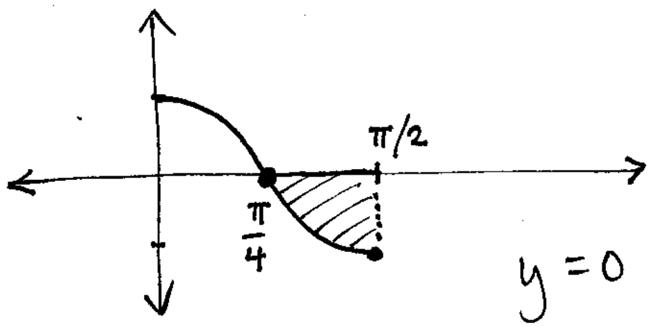
$$\text{Area} = \int_{y=0}^{y=4} \left(\sqrt{y} - \frac{1}{2}y\right) dy = \int_0^4 \left(y^{1/2} - \frac{1}{2}y\right) dy$$

$$= \left[\frac{y^{3/2}}{3/2} - \frac{y^2}{4}\right]_0^4 = \frac{2}{3}\cdot 4^{3/2} - \frac{4^2}{4} = \frac{2}{3}\cdot 2^3 - 4$$

$$= \frac{16}{3} - \frac{12}{3} = \frac{4}{3}$$

4. Sketch the region bounded by the curves, and find its area.

$$y = \cos 2x, \qquad y = 0, \qquad x = \pi/4, \qquad x = \pi/2$$



$$\cos\left(2\cdot\frac{\pi}{4}\right) = \cos\frac{\pi}{2} = 0$$

$$\cos\left(2\cdot\frac{\pi}{2}\right) = \cos\pi = -1$$

$y=0$ is top

$y = \cos 2x$ is bottom

$$\text{Area} = \int_{\pi/4}^{\pi/2} (0 - \cos 2x)\,dx = -\int_{\pi/4}^{\pi/2} \cos 2x\,dx$$

$$= \int_{\pi/2}^{\pi/4} \cos 2x\,dx = \left[\frac{\sin 2x}{2}\right]_{\pi/2}^{\pi/4} = \frac{1}{2}\Big[\sin 2x\Big]_{\pi/2}^{\pi/4}$$

$$= \frac{1}{2}\left(\sin\frac{2\pi}{4} - \sin\frac{2\pi}{2}\right) = \frac{1}{2}\left(\sin\frac{\pi}{2} - \sin\pi\right)$$

$$= \frac{1}{2}(1-0) = \frac{1}{2}$$

5. Sketch the region bounded by the curves, and find its area.

$$y = e^x, \qquad y = e^{2x}, \qquad x = 0, \qquad x = \ln 2$$

If $0 < x < \ln 2$ then x is positive so $x < 2x$

so $e^x < e^{2x}$

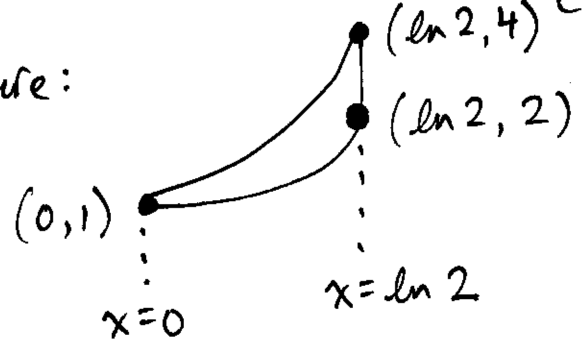
$x = 0 \Rightarrow e^x = e^0 = 1$

$e^{2x} = e^0 = 1$

$x = \ln 2 \Rightarrow e^x = e^{\ln 2} = 2$

$e^{2x} = e^{2\ln 2} = (e^{\ln 2})^2 = 4$

Rough picture:



$$\text{Area} = \int_0^{\ln 2} (e^{2x} - e^x)\, dx = \left[\frac{e^{2x}}{2} - e^x \right]_0^{\ln 2}$$

$$= \frac{1}{2}\left[e^{2x}\right]_0^{\ln 2} - \left[e^x\right]_0^{\ln 2}$$

$$= \frac{1}{2}\left(e^{2\ln 2} - e^0\right) - \left(e^{\ln 2} - e^0\right)$$

$$= \frac{1}{2}(4 - 1) - (2 - 1) = \frac{3}{2} - 1 = \frac{1}{2}$$

6. Sketch the region bounded by the curves, and find its area.

$$x = 1/y, \qquad x = 0, \qquad y = 1, \qquad y = 3$$

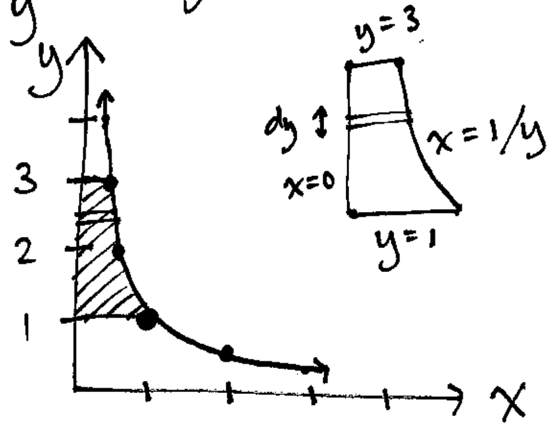
$x = f(y) \Rightarrow$ left curve, right curve

If input y is between 1 and 3

then $x = \frac{1}{y}$ has larger outputs than $x = 0$

So $x = \frac{1}{y}$ is right curve and $x = 0$ is left curve.

Graph:



$$\text{Area} = \int_{y=1}^{y=3} \underbrace{(\text{right} - \text{left})}_{\text{width}} \underbrace{dy}_{\text{height}} = \int_{y=1}^{y=3} \left(\frac{1}{y} - 0\right) dy$$

$$= \int_1^3 \frac{1}{y}\, dy = \Big[\ln|y|\Big]_1^3 = \ln|3| - \ln|1|$$

$$= \ln 3 - \ln 1$$

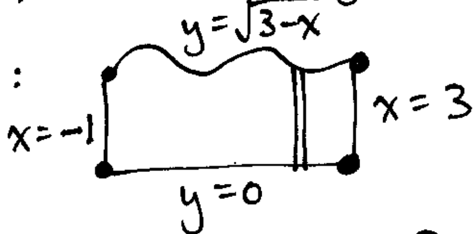
$$= \ln 3 - 0$$

$$= \ln 3$$

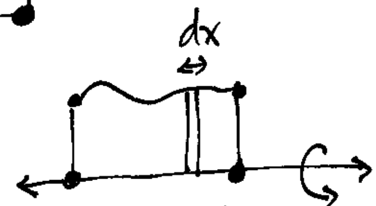
7. Let A denote the region bounded by $y = \sqrt{3-x}$ and the x-axis between $x = -1$ and $x = 3$. Find the volume obtained by revolving A around the x-axis.

One curve is $y = \sqrt{3-x}$. Positive outputs if $x < 3$.
Other curve is x-axis, i.e. $y = 0$.
$y = \sqrt{3-x}$ is top curve and $y = 0$ is bottom curve.

"Lying" picture:



Revolve around x-axis:



Slice perpendicular to axis of rev
WASHERS

Washers (where inner radius happens to be 0)

$$\text{Volume} = \int_{x=-1}^{x=3} \left(\pi R^2 - \pi r^2\right) dx = \pi \int_{-1}^{3} \left(R^2 - r^2\right) dx$$

$R = (\text{top curve}) - (\text{axis of rev.}) = \sqrt{3-x} - 0 = \sqrt{3-x}$

$r = (\text{bottom curve}) - (\text{axis of rev.}) = 0 - 0 = 0$

$$\text{Volume} = \pi \int_{-1}^{3} R^2 \, dx = \pi \int_{-1}^{3} \left(\sqrt{3-x}\right)^2 dx$$

$$= \pi \int_{-1}^{3} (3-x)\, dx = \pi \left[3x - \frac{x^2}{2} \right]_{-1}^{3}$$

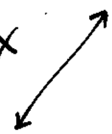
$$= \pi \left(3[x]_{-1}^{3} - \frac{1}{2}[x^2]_{-1}^{3} \right) = \pi \left(3 \cdot 4 - \frac{1}{2}(9-1) \right) = 8\pi$$

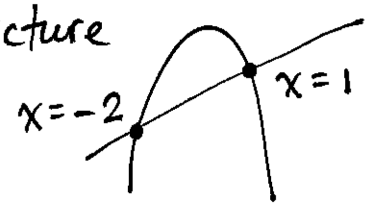
8. Let A denote the region bounded by $y = x$ and $y = 2 - x^2$ between $x = 0$ and $x = 1$. Find the volume obtained by revolving A around the x-axis.

Intersection points? $x = 2 - x^2$

$x^2 + x - 2 = 0 \qquad (x+2)(x-1) = 0$

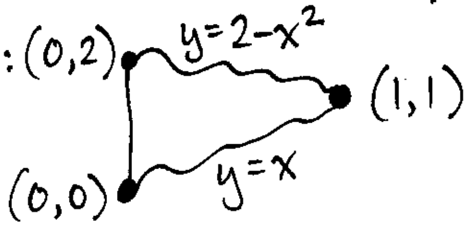
$x = -2, \quad x = 1$

$y = x$  $y = 2 - x^2$  Rough picture

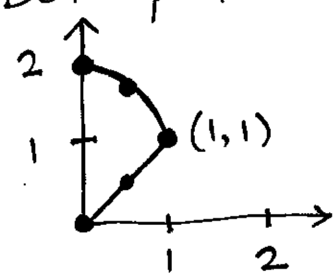


If $0 \le x \le 1$ then $2 - x^2$ is top curve, x is bottom

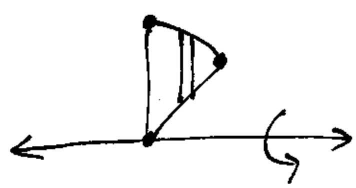
"Lying" picture:



Better picture:



$y = f(x) \Rightarrow$  dx. Revolve around x-axis



WASHERS

$$\text{Volume} = \int_{x=0}^{x=1} \left(\pi R^2 - \pi r^2\right) dx = \pi \int_0^1 \left(R^2 - r^2\right) dx$$

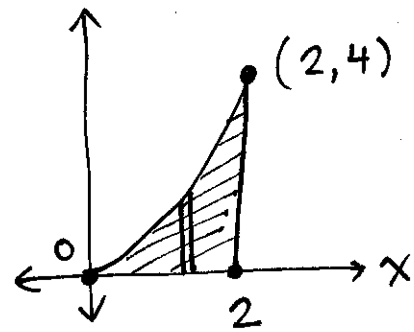
$R = \text{top} - \text{axis} = 2 - x^2, \quad r = \text{bottom} - \text{axis} = x$

$$\pi \int_0^1 \left((2-x^2)^2 - x^2\right) dx = \pi \int_0^1 \left(4 - 4x^2 + x^4 - x^2\right) dx$$

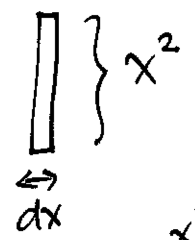
$$= \pi \int_0^1 \left(4 - 5x^2 + x^4\right) dx = \pi \left[4x - \frac{5x^3}{3} + \frac{x^5}{5}\right]_0^1 = \ldots = \frac{38\pi}{15}$$

9. Find the volume of the solid whose base is the region bounded by $y = x^2$ and the x-axis between $x = 0$ and $x = 2$, and whose cross-sections perpendicular to the x-axis are squares.

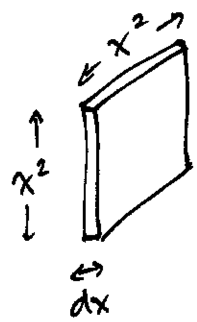
Sketch of region:



Typical slice of <u>area</u> :



Typical slice of <u>volume</u>:



$$\text{Total volume} = \int_{x=0}^{x=2} (\text{volume of slice}) = \int_0^2 x^2 \cdot x^2 \cdot dx$$

$$= \int_0^2 x^4\, dx = \left[\frac{x^5}{5} \right]_0^2 = \frac{32}{5}$$

10. Find the volume of the solid that results when the region enclosed by $y = \sqrt{25 - x^2}$ and $y = 3$ is revolved around the x-axis.

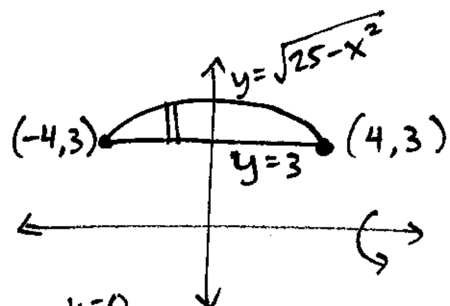
Intersection points? $\sqrt{25-x^2} = 3$

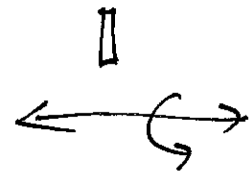
$$25 - x^2 = 9 \qquad x^2 = 16$$
$$x = \pm 4$$

"Test" x value between $x=-4$ and $x=4$: Say $x=0$

$\underbrace{y=\sqrt{25-x^2}}_{\text{top}}$ has bigger outputs than $\underbrace{y=3}_{\text{bottom}}$

Rough picture:





WASHERS

$$R = (\overset{y=\sqrt{25-x^2}}{\text{top curve}}) - (\overset{y=0}{\text{axis of rev.}}) = \sqrt{25-x^2}$$
$$r = (\underset{y=3}{\text{bottom curve}}) - (\underset{y=0}{\text{axis of rev.}}) = 3$$

$$\text{Volume} = \pi \int_{x=-4}^{x=4} (R^2 - r^2)\,dx = \pi \int_{-4}^{4} (25 - x^2 - 9)\,dx$$

$$= \pi \int_{-4}^{4} (16 - x^2)\,dx = \pi \left[16x - \frac{x^3}{3}\right]_{-4}^{4}$$

$$= \pi \left(16[x]_{-4}^{4} - \frac{1}{3}[x^3]_{-4}^{4} \right) = \pi \left(16(4-(-4)) - \frac{1}{3}(64-(-64)) \right)$$

$$= \pi \left(16 \cdot 8 - \frac{128}{3} \right) = \ldots = \frac{256\pi}{3}$$

11. Find the volume of the solid that results when the region enclosed by the given curves is revolved around the y-axis.

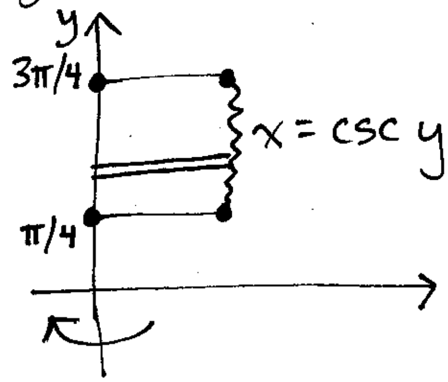
$$x = \csc y, \qquad y = \pi/4, \qquad y = 3\pi/4, \qquad x = 0$$

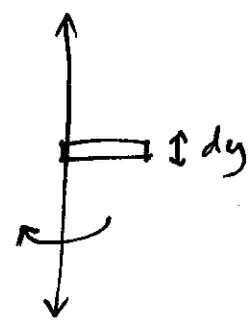
$x = f(y)$. We don't use csc much, but $\csc y = \dfrac{1}{\sin y}$

If $\dfrac{\pi}{4} \le y \le \dfrac{3\pi}{4}$ then $\sin y$ is positive so $\csc y$ is positive

$x = \csc y$ is "right curve" and $x = 0$ is "left curve"

"Lying" picture:





WASHERS

(disks in this case)

$$\text{Volume} = \int_{y=\pi/4}^{y=3\pi/4} \pi R^2 \, dy \qquad (r = 0)$$

$$R = \underset{x = \csc y}{(\text{right curve})} - \underset{x=0}{(\text{axis of rev.})} = \csc y$$

$$\text{Volume} = \pi \int_{\pi/4}^{3\pi/4} \csc^2 y \, dy = \pi \Big[-\cot y \Big]_{\pi/4}^{3\pi/4}$$

$$= \pi \Big[\cot y \Big]_{3\pi/4}^{\pi/4} = \pi \left(\cot \frac{\pi}{4} - \cot \frac{3\pi}{4} \right) = \pi(1 - (-1))$$

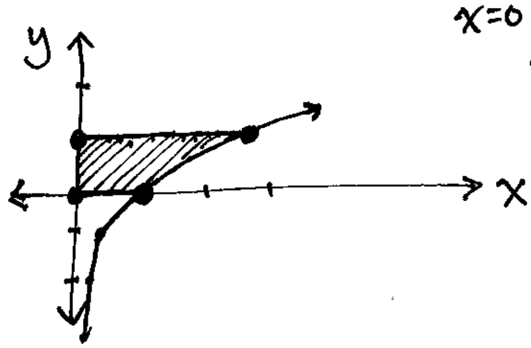
$$= 2\pi$$

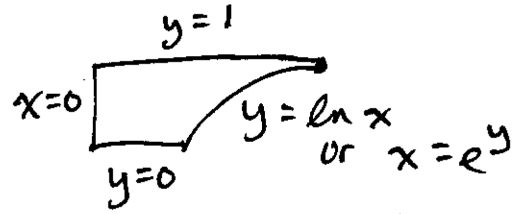
12. Find the volume of the solid that results when the region enclosed by the given curves is revolved around the y-axis.

$$y = \ln x, \qquad x = 0, \qquad y = 0, \qquad y = 1$$

$y = \ln x \;\downarrow\; x = e^y$

Picture:





Easier to use "left curve" and "right curve"

Left curve: $x=0$. Right curve: $x=e^y$.

Disks. (washers with $r=0$)

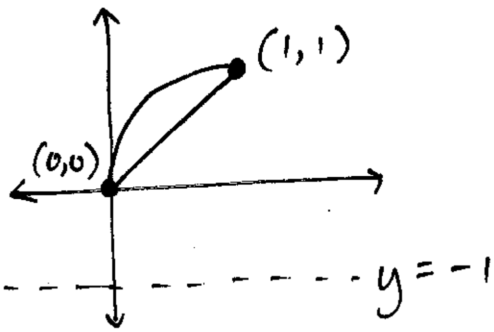
$$\text{Volume} = \int_{y=0}^{y=1} \pi R^2 \, dy \qquad R = \underset{x=e^y}{(\text{right curve})} - \underset{x=0}{(\text{axis of rev.})} = e^y - 0 = e^y$$

$$\pi \int_0^1 (e^y)^2 \, dy = \pi \int_0^1 e^{2y} \, dy = \pi \left[\frac{e^{2y}}{2} \right]_0^1$$

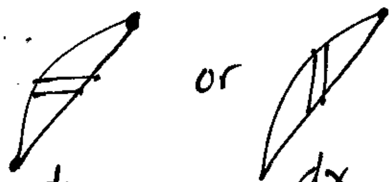
$$= \frac{\pi}{2} \left[e^{2y} \right]_0^1 = \frac{\pi}{2} \left(e^2 - 1 \right)$$

13. Find the volume of the solid that results when the region enclosed by $x = y^2$ and $x = y$ is revolved around the line $y = -1$.

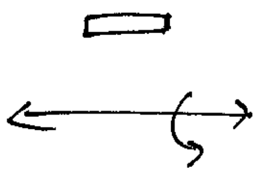
Rough picture:



Can do <u>either</u> ..



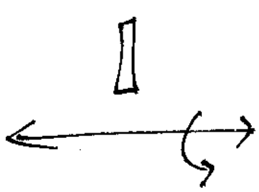
dy	dx
$x = y^2$ left	$y = \sqrt{x}$ top
$x = y$ right	$y = x$ bottom



Shells: $r = \text{slice} - \text{axis} = y - (-1) = y + 1$

$h = \text{right} - \text{left} = y - y^2$

$$\text{Volume} = \int_0^1 2\pi r h \, dy = 2\pi \int_0^1 \underbrace{(y+1)(y-y^2)}_{\text{expand}} \, dy$$



Washers: $R = \text{top} - \text{axis} = \sqrt{x} - (-1) = \sqrt{x} + 1$

$r = \text{bottom} - \text{axis} = x - (-1) = x + 1$

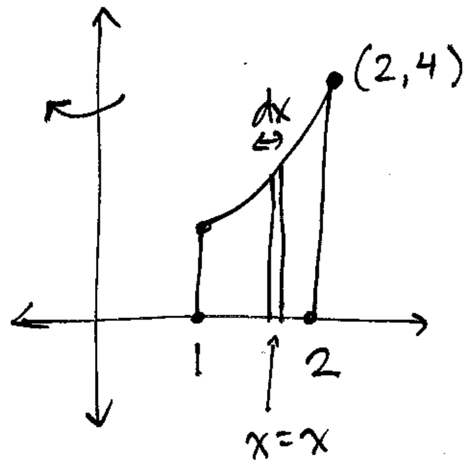
$$\text{Volume} = \int_0^1 (\pi R^2 - \pi r^2) \, dx = \pi \int_0^1 \left(\underbrace{(\sqrt{x}+1)^2}_{\text{expand}} - \underbrace{(x+1)^2}_{\text{expand}} \right) dx$$

Both ways take several steps. dy may be slightly faster.

Either way I get volume $= \dfrac{\pi}{2}$.

14. Let A denote the region bounded by $y = x^2$ and the x-axis between $x = 1$ and $x = 2$. Find the volume obtained by revolving A around the y-axis.

Rough picture:



Cylindrical shells.

$r = \underset{x=x}{(\text{typical slice})} - \underset{x=0}{(\text{axis of rev.})} = x - 0 = x$

$h = \underset{y=x^2}{(\text{top curve})} - \underset{y=0}{(\text{bottom curve})} = x^2 - 0 = x^2$

$$\text{Volume} = \int_{x=1}^{x=2} 2\pi r h \, dx = 2\pi \int_1^2 r h \, dx$$

$$= 2\pi \int_1^2 x \cdot x^2 \, dx = 2\pi \int_1^2 x^3 \, dx = 2\pi \left[\frac{x^4}{4} \right]_1^2$$

$$= \frac{2\pi}{4} \left[x^4 \right]_1^2 = \frac{\pi}{2}\left(2^4 - 1^4\right) = \frac{\pi}{2}(16-1) = \frac{15\pi}{2}$$

$x = 0$

15. Let A denote the region bounded by $x = 2y - 2y^2$ and the y-axis. Find the volume obtained by revolving A around the x-axis.

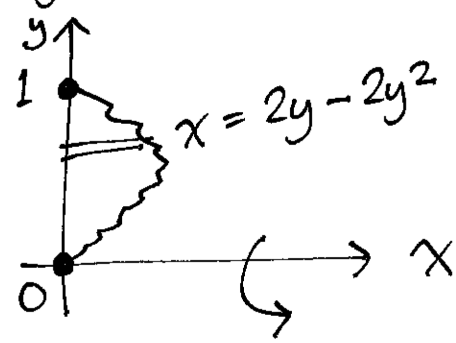
Intersection points? $2y - 2y^2 = 0$

$2y(1-y) = 0 \quad y = 0,\ y = 1$

If $y = \frac{1}{2}$, say, then $2y - 2y^2 = 2y(1-y)$ is positive

$x = 2y - 2y^2$ is "right curve" and $x = 0$ is "left curve"

"Lying" picture:



Cylindrical shells. Volume $= \displaystyle\int_{y=0}^{y=1} 2\pi r h \, dy = 2\pi \int_0^1 r h \, dy$

$r = \underset{y=y}{(\text{typical slice})} - \underset{y=0}{(\text{axis of rev.})} = y - 0 = y$

$h = \underset{x=2y-2y^2}{(\text{right curve})} - \underset{x=0}{(\text{left curve})} = 2y - 2y^2 - 0 = 2y - 2y^2$

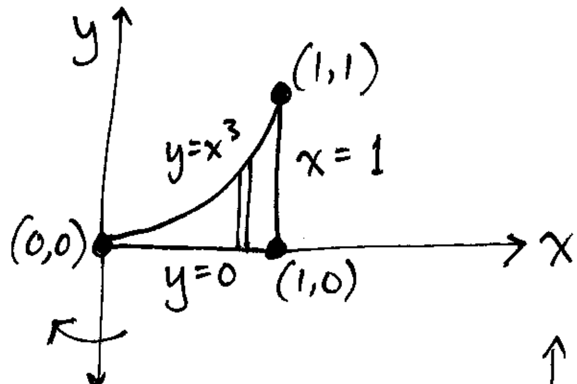
$$\text{Volume} = 2\pi \int_0^1 y(2y - 2y^2)\,dy = 2\pi \int_0^1 (2y^2 - 2y^3)\,dy$$

$$= 2\pi \left[\frac{2y^3}{3} - \frac{2y^4}{4} \right]_0^1 = 2\pi \left(\frac{2}{3} - \frac{1}{2} \right) = \frac{2\pi}{6} = \frac{\pi}{3}$$

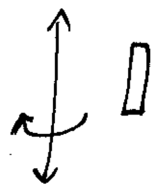
16. Find the volume of the solid that results when the region enclosed by the given curves is revolved around the y-axis.

$$y = x^3, \qquad x = 1, \qquad y = 0$$

Rough picture:



If we keep as $y = f(x)$ then SHELLS



$$\text{Volume} = \int_{x=0}^{x=1} 2\pi r h \, dx = 2\pi \int_0^1 r h \, dx$$

$$r = \underset{x=x}{(\text{typical slice})} - \underset{x=0}{(\text{axis. of rev.})} = x - 0 = x$$

$$h = \underset{y=x^3}{(\text{top curve})} - \underset{y=0}{(\text{bottom curve})} = x^3 - 0 = x^3$$

$$2\pi \int_0^1 x \cdot x^3 \, dx = 2\pi \int_0^1 x^4 \, dx = 2\pi \left[\frac{x^5}{5} \right]_0^1$$

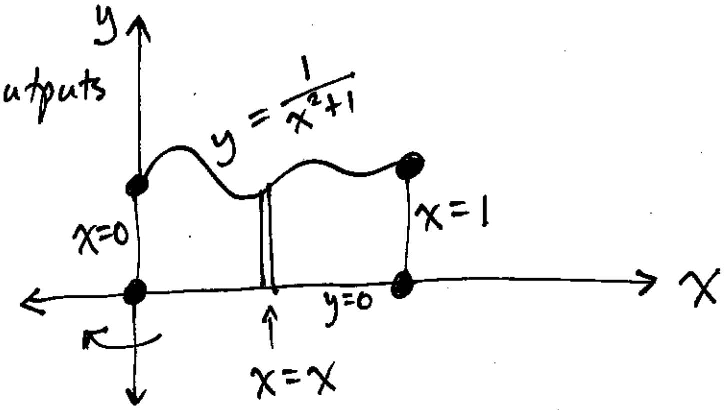
$$= \frac{2\pi}{5}$$

17. Find the volume of the solid that results when the region enclosed by the given curves is revolved around the y-axis.

$$y = \frac{1}{x^2+1}, \qquad x = 0, \qquad x = 1, \qquad y = 0$$

$\downarrow$ positive outputs

"Lying" picture:



Shells.

$$\text{Volume} = \int_{x=0}^{x=1} 2\pi r h \, dx = 2\pi \int_0^1 r h \, dx$$

$$r = \underset{x=x}{(\text{typical slice})} - \underset{x=0}{(\text{axis of rev.})} = x - 0 = x$$

$$h = \underset{y=\frac{1}{x^2+1}}{(\text{top curve})} - \underset{y=0}{(\text{bottom curve})} = \frac{1}{x^2+1} - 0 = \frac{1}{x^2+1}$$

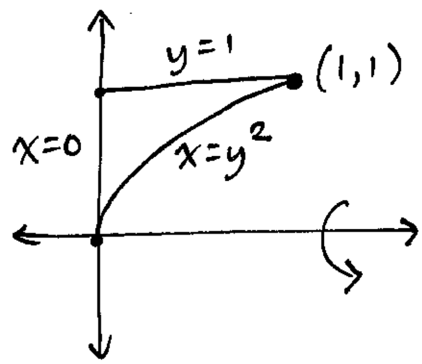
$$2\pi \int_0^1 x \cdot \frac{1}{x^2+1} \, dx = \pi \int_0^1 \frac{2x}{x^2+1} \, dx \qquad \text{Sub } u = x^2+1, \; du = 2x\,dx$$

$$= \pi \int_{u=1}^{u=2} \frac{1}{u} \, du = \pi \Big[\ln|u| \Big]_{u=1}^{u=2} = \pi \ln 2$$

18. Find the volume of the solid that results when the region enclosed by the given curves is revolved around the x-axis.

$$y^2 = x, \qquad y = 1, \qquad x = 0$$

Rough picture:



Two possibilities:

$\Rightarrow$ top is $y=1$, bottom is $y=\sqrt{x}$. Washers. $R=1$

$r=\sqrt{x}$

dx

$$\text{Volume} = \pi\int_0^1 \left(1^2-(\sqrt{x})^2\right)dx = \pi\int_0^1 (1-x)\,dx$$

$$= \pi\left[x-\frac{x^2}{2}\right]_0^1 = \frac{\pi}{2}.$$

dy $\Rightarrow$ right is $x=y^2$, left is $x=0$. Shells. $r = y-0$

$h = y^2-0$

$$\text{Volume} = 2\pi\int_0^1 y\cdot y^2\,dy = 2\pi\int_0^1 y^3\,dy = 2\pi\left[\frac{y^4}{4}\right]_0^1$$

$$= 2\pi\cdot\frac{1}{4} = \frac{\pi}{2}.$$

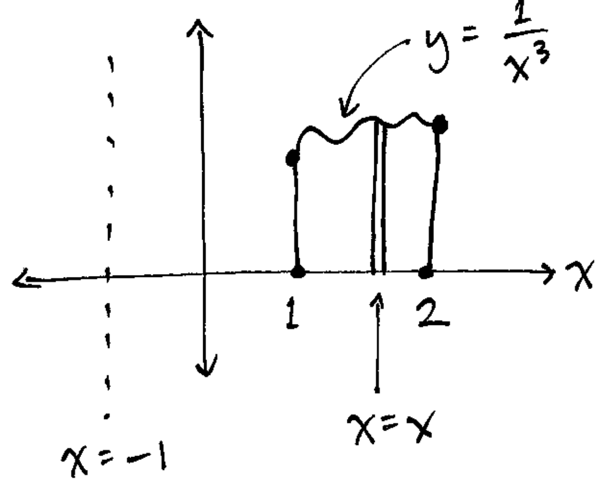
19. Find the volume of the solid that is generated when the region enclosed by

$$y = \frac{1}{x^3}, \qquad x = 1, \qquad x = 2, \qquad y = 0$$

is revolved around the line $x = -1$.

If $1 \le x \le 2$ then $y = \dfrac{1}{x^3}$ has positive outputs.

"Lying" picture:



Shells: $r = (\text{typical slice}) - (\text{axis of rev.}) = x - (-1) = x + 1$

$h = \underbrace{(\text{top curve})}_{y = \frac{1}{x^3}} - \underbrace{(\text{bottom curve})}_{y=0} = \dfrac{1}{x^3}$

$$\text{Volume} = 2\pi \int_1^2 r h \, dx = 2\pi \int_1^2 \frac{x+1}{x^3} \, dx$$

$$= 2\pi \int_1^2 \left(\frac{1}{x^2} + \frac{1}{x^3} \right) dx = 2\pi \int_1^2 \left(x^{-2} + x^{-3} \right) dx$$

$$= 2\pi \left[\frac{x^{-1}}{-1} + \frac{x^{-2}}{-2} \right]_1^2 = 2\pi \left[-\frac{1}{x} - \frac{1}{2x^2} \right]_1^2$$

$$= 2\pi \left[\frac{1}{x} + \frac{1}{2x^2} \right]_2^1 = 2\pi \left(1 + \frac{1}{2} - \frac{1}{2} - \frac{1}{8} \right) = 7\pi/4$$

20. Find the length of the curve $y = 3x^{3/2} - 1$ between $x = 0$ and $x = 1$.

$$\frac{dy}{dx} = 3 \cdot \frac{3}{2} x^{1/2} = \frac{9}{2} x^{1/2}$$

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \sqrt{1 + \left(\frac{9}{2} x^{1/2}\right)^2}\, dx = \sqrt{1 + \frac{81x}{4}}\, dx$$

$$\text{Length} = \int_{x=0}^{x=1} ds = \int_0^1 \sqrt{1 + \frac{81x}{4}}\, dx$$

Sub $u = 1 + \frac{81x}{4} \Rightarrow du = \frac{81}{4} dx \Rightarrow \frac{4}{81} du = dx$

$x = 0 \Rightarrow u = 1$, $x = 1 \Rightarrow u = 1 + \frac{81}{4} = 85/4$

$$\int_{u=1}^{u=85/4} \sqrt{u} \cdot \frac{4}{81}\, du = \frac{4}{81} \int_{u=1}^{u=85/4} u^{1/2}\, du$$

$$= \frac{4}{81} \left[\frac{2}{3} u^{3/2} \right]_{u=1}^{u=85/4} = \frac{8}{243} \left[u^{3/2} \right]_1^{85/4}$$

$$= \frac{8}{243} \left(\left(\frac{85}{4}\right)^{3/2} - 1 \right) \quad \text{or} \quad \frac{8}{243} \left(\frac{85^{3/2}}{8} - \frac{8}{8} \right) = \frac{85^{3/2} - 8}{243}$$

21. Find the area of the surface generated by revolving the given curve around the x-axis.

$$y = 7x, \qquad 0 \le x \le 1$$

$$\text{Surface area} = \int 2\pi r\, ds = \int 2\pi f(x)\sqrt{1+(f'(x))^2}\, dx$$

$$= 2\pi \int_0^1 7x \underbrace{\sqrt{1+7^2}}_{\sqrt{50}}\, dx = 14\pi\sqrt{50}\int_0^1 x\, dx$$

$$= 14\pi\sqrt{50}\left[\frac{x^2}{2}\right]_0^1 = 7\pi\sqrt{50}$$

$$\text{or } 7\pi\sqrt{25}\cdot\sqrt{2}$$

$$= 35\pi\sqrt{2}$$

22. Find the area of the surface generated by revolving the given curve around the x-axis.

$$y = \sqrt{4 - x^2}, \qquad -1 \le x \le 1$$

$$f(x) = (4 - x^2)^{1/2}$$

$$f'(x) = \frac{1}{2}(4 - x^2)^{-1/2} \cdot (-2x)$$

$$= -x(4 - x^2)^{-1/2} = \frac{-x}{\sqrt{4 - x^2}}$$

$$ds = \sqrt{1 + (f'(x))^2}\, dx = \sqrt{1 + \frac{x^2}{4 - x^2}}\, dx$$

$$= \sqrt{\frac{4 - x^2}{4 - x^2} + \frac{x^2}{4 - x^2}}\, dx = \sqrt{\frac{4}{4 - x^2}}\, dx = \frac{2}{\sqrt{4 - x^2}}\, dx$$

$$\text{Surface area} = \int_{-1}^{1} 2\pi r\, ds = \int_{-1}^{1} 2\pi f(x)\, ds$$

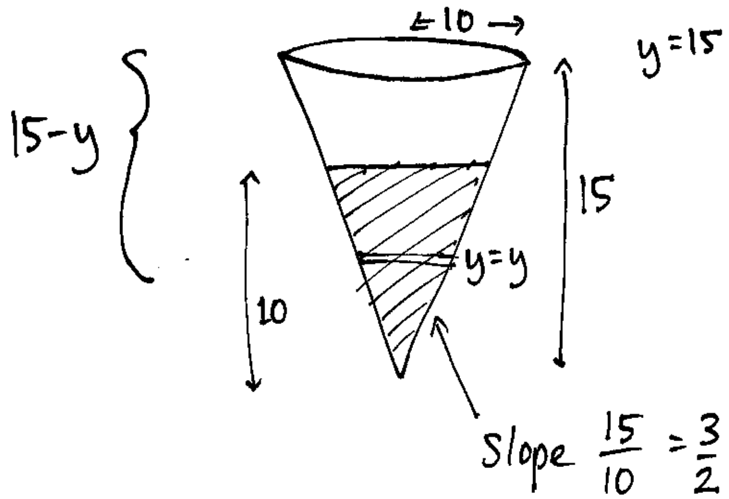
$$= 2\pi \int_{-1}^{1} \sqrt{4 - x^2} \cdot \frac{2}{\sqrt{4 - x^2}}\, dx = 2\pi \int_{-1}^{1} 2\, dx$$

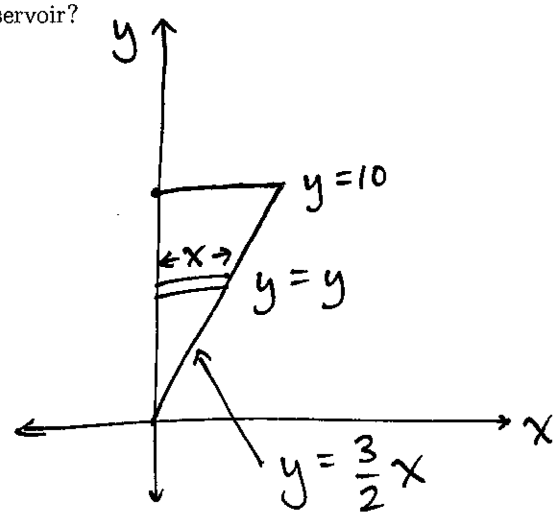
$$= 2\pi \cdot 2 \cdot (1 - (-1)) = 8\pi$$

If you're curious, density of water is $\rho = 62.4$ pounds per cubic foot.

If using feet and pounds, work will be in foot-pounds

23. A cone-shaped water reservoir is 20 ft in diameter across the top and 15 ft deep. If the reservoir is filled to a depth of 10 ft, how much work is required to pump all the water to the top of the reservoir?





"Typical slice" located at $y=y$ must be lifted a distance $15-y$

Work of lifting one slice = (weight of slice) $\cdot$ (distance slice is lifted)

$$= \underbrace{\text{density}}_{\rho} \cdot (\text{volume of slice}) \cdot (15-y) = \rho \cdot \pi x^2 \, dy \cdot (15-y)$$

$$x = \frac{2}{3} y$$

Element of work: $\rho \cdot \pi \left(\frac{2}{3} y\right)^2 dy \, (15-y)$

$$= \rho \cdot \frac{4\pi}{9} \cdot y^2 (15-y) \, dy. \quad \text{Work} = \int_{y=0}^{y=10} \rho \cdot \frac{4\pi}{9} y^2 (15-y) \, dy$$

$$= \rho \cdot \frac{4\pi}{9} \int_0^{10} (15y^2 - y^3) \, dy = \rho \cdot \frac{4\pi}{9} \left[15\frac{y^3}{3} - \frac{y^4}{4} \right]_0^{10}$$

$$= \rho \cdot \frac{4\pi}{9} (5000 - 2500) = \rho \cdot \frac{10000\pi}{9}$$