MAC2312

MORE suggested problems on Chapter 7 material (techniques of integration)

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1. Evaluate the integral.

$$\int \frac{1}{x^2 - 3x - 4}\, dx$$

$x^2-3x-4 = (x+1)(x-4)$

$$\frac{1}{x^2-3x-4} = \frac{1}{(x+1)(x-4)} = \frac{A}{x+1} + \frac{B}{x-4}$$

multiply both sides by $(x+1)(x-4)$

$$1 = A(x-4) + B(x+1)$$

Could sub in specific x values. $x=4 \Rightarrow 1 = 5B \Rightarrow B = \frac{1}{5}$

$x=-1 \Rightarrow 1 = -5A \Rightarrow A = -\frac{1}{5}$

$$\int \frac{1}{x^2-3x-4}\, dx = \int \left(\frac{-1/5}{x+1} + \frac{1/5}{x-4} \right) dx$$

$$= -\frac{1}{5}\int \frac{1}{x+1}\, dx + \frac{1}{5}\int \frac{1}{x-4}\, dx$$

$$= -\frac{1}{5} \ln|x+1| + \frac{1}{5}\ln|x-4| + C$$

2. Evaluate the integral.

$$\int \frac{1}{x(x^2-1)}\,dx$$

$$\frac{1}{x(x^2-1)} = \frac{1}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$$

$$1 = A(x+1)(x-1) + Bx(x-1) + Cx(x+1)$$

Sub $x=0$: $1 = A\cdot 1\cdot(-1) + 0 + 0 \Rightarrow 1 = -A \Rightarrow A = -1$

Sub $x=1$: $1 = 0 + 0 + C\cdot 1\cdot 2 \Rightarrow 1 = 2C \Rightarrow C = \frac{1}{2}$

Sub $x=-1$: $1 = 0 + B\cdot(-1)(-2) + 0 \Rightarrow 1 = 2B \Rightarrow B = \frac{1}{2}$

$$\int \frac{1}{x(x^2-1)}\,dx = \int \left(\frac{-1}{x} + \frac{1/2}{x+1} + \frac{1/2}{x-1} \right) dx$$

$$= -\int \frac{1}{x}\,dx + \frac{1}{2}\int \frac{1}{x+1}\,dx + \frac{1}{2}\int \frac{1}{x-1}\,dx$$

$$= -\ln|x| + \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C$$

3. Evaluate the integral.

$$\int \frac{x^2 - 8}{x+3}\, dx$$

One possible method: Long division

$$\begin{array}{r} x-3 \\ x+3 \overline{) x^2 + 0x - 8} \\ \underline{x^2 + 3x} \\ -3x - 8 \\ \underline{-3x - 9} \\ 1 \end{array}$$

Another possible method: Clever rearranging

$$\int \frac{x^2 - 9 + 1}{x+3}\, dx = \int \left(\frac{x^2-9}{x+3} + \frac{1}{x+3} \right) dx$$

$$= \int \left(x - 3 + \frac{1}{x+3} \right) dx$$

$$= \frac{x^2}{2} - 3x + \ln|x+3| + C$$

4. Evaluate the integral.

$$\int \frac{2x^2-1}{(4x-1)(x^2+1)}\,dx$$

$$\frac{2x^2-1}{(4x-1)(x^2+1)} = \frac{A}{4x-1} + \frac{Bx+C}{x^2+1}$$

Multiply both sides by $(4x-1)(x^2+1)$

$$2x^2-1 = A(x^2+1) + (Bx+C)(4x-1)$$

$$2x^2+0x+(-1) = Ax^2+A+4Bx^2-Bx+4Cx-C$$

$$= (A+4B)x^2 + (4C-B)x + (A-C)$$

$$\Rightarrow \quad A+4B=2$$

$$4C-B=0 \Rightarrow 4C-B=0$$

$$A-C=-1 \Rightarrow 4A-4C=-4$$

$$\overline{4A-B=-4}$$

$$16A-4B=-16$$

$$A+4B=2$$

$$\overline{17A = -14}$$

$$A=\frac{-14}{17}$$

Then $\frac{-14}{17}-C=-1 \Rightarrow \frac{-14}{17}-C=\frac{-17}{17} \Rightarrow C=\frac{3}{17} \Rightarrow B=\frac{12}{17}$

$$\int \frac{2x^2-1}{(4x-1)(x^2+1)}\,dx = \int\left(\frac{\frac{-14}{17}}{4x-1} + \frac{\frac{12}{17}x+\frac{3}{17}}{x^2+1}\right)dx$$

$$= -\frac{14}{17}\int\frac{1}{4x-1}\,dx + \frac{12}{17}\int\frac{x}{x^2+1}\,dx + \frac{3}{17}\int\frac{1}{x^2+1}\,dx$$

$$= -\frac{14}{17}\cdot\frac{1}{4}\ln|4x-1| + \frac{12}{17}\cdot\frac{1}{2}\ln(x^2+1) + \frac{3}{17}\arctan x + C$$

5. Evaluate the integral by making a substitution that converts the integrand to a rational function.

$$\int \frac{\cos\theta}{\sin^2\theta + 4\sin\theta - 5}\, d\theta \quad = I$$

Sub $u = \sin\theta$

$\Rightarrow du = \cos\theta\, d\theta$

$$I = \int \frac{1}{u^2+4u-5}\, du = \int \frac{1}{(u+5)(u-1)}\, du$$

Next, do partial fractions. OR, use a clever trick:

$$\frac{1}{6}\int \frac{6}{(u+5)(u-1)}\, du = \frac{1}{6}\int \frac{(u+5)-(u-1)}{(u+5)(u-1)}\, du$$

$$= \frac{1}{6}\int \left(\frac{u+5}{(u+5)(u-1)} - \frac{u-1}{(u+5)(u-1)} \right) du$$

$$= \frac{1}{6}\int \left(\frac{1}{u-1} - \frac{1}{u+5} \right) du$$

Note: This should agree with what you get using partial fractions

$$= \frac{1}{6}\left(\ln|u-1| - \ln|u+5| \right)$$

$$= \frac{1}{6}\left(\ln|\sin\theta - 1| - \ln|\sin\theta + 5| \right)$$

6. Use both the Trapezoid Rule and Simpson's Rule with $n = 4$ to estimate the definite integral.

$$\int_0^4 \frac{1}{x^3+1}\,dx$$

$a=0,\ b=4.\quad n=4 \Rightarrow \Delta x = \frac{b-a}{n} = \frac{4-0}{4} = 1$

$x_0 = 0,\ x_1 = 1,\ x_2 = 2,\ x_3 = 3,\ x_4 = 4$

x	x^3	x^3+1	$f(x) = \frac{1}{x^3+1}$
0	0	1	1
1	1	2	1/2
2	8	9	1/9
3	27	28	1/28
4	64	65	1/65

Trapezoid: $\frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right)$

$= \frac{1}{2}\left(1 + 2\cdot\frac{1}{2} + 2\cdot\frac{1}{9} + 2\cdot\frac{1}{28} + \frac{1}{65}\right)$

$= \ldots = \frac{18911}{16380} = 1.1545$

Simpson: $\frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right)$

$= \frac{1}{3}\left(1 + 4\cdot\frac{1}{2} + 2\cdot\frac{1}{9} + 4\cdot\frac{1}{28} + \frac{1}{65}\right)$

$= \ldots = \frac{13843}{12285} = 1.1268$

7. Use the Trapezoid Rule and Simpson's Rule to estimate the integral, using $n = 2$ subintervals.

$$\int_1^{49} \frac{1}{\sqrt{x}+1}\,dx$$

$a=1,\ b=49,\ n=2$

Can you also evaluate the integral exactly?

$\Delta x = \frac{b-a}{n} = \frac{49-1}{2} = 24$

$x_0 = 1,\ x_1 = 25,\ x_2 = 49$

$f(x_0) = \frac{1}{\sqrt{1}+1} = \frac{1}{2},\ f(x_1) = \frac{1}{\sqrt{25}+1} = \frac{1}{6},\ f(x_2) = \frac{1}{\sqrt{49}+1} = \frac{1}{8}$

Trapezoid: $\frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + f(x_2)\right) = \frac{24}{2}\left(\frac{1}{2} + 2\cdot\frac{1}{6} + \frac{1}{8}\right)$

$= 12\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{8}\right) = 12\left(\frac{12}{24} + \frac{8}{24} + \frac{3}{24}\right) = 12\cdot\frac{23}{24} = \frac{23}{2} = 11.5$

Simpson: $\frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right) = \frac{24}{3}\left(\frac{1}{2} + 4\cdot\frac{1}{6} + \frac{1}{8}\right)$

$= 8\left(\frac{1}{2} + \frac{2}{3} + \frac{1}{8}\right) = 8\left(\frac{12}{24} + \frac{16}{24} + \frac{3}{24}\right) = 8\cdot\frac{31}{24} = \frac{31}{3} = 10.333\ldots$

Exact? Sub $u = \sqrt{x}+1 = x^{1/2}+1 \Rightarrow du = \frac{1}{2}x^{-1/2}dx = \frac{1}{2\sqrt{x}}dx$

$x=1 \Rightarrow u=\sqrt{1}+1 = 2$

$x=49 \Rightarrow u=\sqrt{49}+1 = 8$

Integral $= \int_{x=1}^{49} \frac{2\sqrt{x}}{\sqrt{x}+1}\cdot\frac{1}{2\sqrt{x}}\,dx$

$= \int_{u=2}^{u=8} \frac{2(u-1)}{u}\,du = 2\int_2^8 \frac{u-1}{u}\,du = 2\int_2^8 \left(\frac{u}{u} - \frac{1}{u}\right)du$

$= 2\int_2^8 \left(1 - \frac{1}{u}\right)du = 2\left[u - \ln u\right]_2^8 = 2\left((8-\ln 8) - (2-\ln 2)\right)$

$= 2(8 - 3\ln 2 - 2 + \ln 2) = 2(6 - 2\ln 2) = 12 - 4\ln 2$

8. Determine whether the integral converges or diverges, and find its value if it converges.

$$\int_3^\infty \frac{2}{x^2-1}\,dx$$

Consider $\int_3^M \frac{2}{x^2-1}\,dx = \int_3^M \frac{2}{(x+1)(x-1)}\,dx$ Could use partial fractions

$$= \int_3^M \frac{(x+1)-(x-1)}{(x+1)(x-1)}\,dx = \int_3^M \left(\frac{x+1}{(x+1)(x-1)} - \frac{x-1}{(x+1)(x-1)}\right)dx$$

$$= \int_3^M \left(\frac{1}{x-1} - \frac{1}{x+1}\right)dx = \Big[\ln|x-1| - \ln|x+1|\Big]_3^M$$

$$= \Big(\ln|M-1| - \ln|M+1|\Big) - \Big(\ln|3-1| - \ln|3+1|\Big)$$

$$= \ln|M-1| - \ln|M+1| - \ln 2 + \ln 4$$

$$= \ln|M-1| - \ln|M+1| - \ln 2 + 2\ln 2$$

$$= \ln\left|\frac{M-1}{M+1}\right| + \ln 2.$$ Now let $M \to \infty$. Then $\frac{M-1}{M+1} \to 1$

Answer: $\ln|1| + \ln 2 = 0 + \ln 2 = \ln 2$

9. Determine whether the integral converges or diverges, and find its value if it converges.

$$\int_e^\infty \frac{1}{x \ln^3 x}\, dx$$

Consider $\int_e^M \frac{1}{x \ln^3 x}\, dx$. Sub $u = \ln x$

$du = \frac{1}{x}\, dx$

$x = e \Rightarrow u = \ln e = 1$

$x = M \Rightarrow u = \ln M$

$$\int_{u=1}^{u=\ln M} \frac{1}{u^3}\, du = \int_1^{\ln M} u^{-3}\, du$$

$$= \left[\frac{u^{-2}}{-2} \right]_1^{\ln M} = \left[-\frac{1}{2u^2} \right]_1^{\ln M} = \frac{1}{2}\left[\frac{1}{u^2} \right]_{\ln M}^{1}$$

$$= \frac{1}{2}\left(1 - \frac{1}{(\ln M)^2} \right)$$

Now let $M \to \infty$

Then also $\ln M \to \infty$

Answer: $\frac{1}{2}(1-0) = \frac{1}{2}$

10. Determine whether the integral converges or diverges, and find its value if it converges.

$$\int_0^4 \frac{1}{(x-4)^2}\, dx$$

$x=4$ is "problem" point

Consider $\displaystyle\int_0^M \frac{1}{(x-4)^2}\, dx = \int_0^M (x-4)^{-2}\, dx$ At the <u>end</u>, let $M \to 4$

Could do substitution $u = x-4$ but that's an "easy" substitution

$$\left[\frac{(x-4)^{-1}}{-1} \right]_{x=0}^{x=M} = \left[-\frac{1}{x-4} \right]_0^M = \left[\frac{1}{x-4} \right]_M^0$$

$$= \frac{1}{0-4} - \frac{1}{M-4} = -\frac{1}{4} - \frac{1}{M-4}$$

When $M \to 4$,
this does not approach a limit.

The integral DIVERGES.

11. Determine whether the integral converges or diverges, and find its value if it converges.

$$\int_0^{\pi/2} \tan x \, dx$$

Remember $\tan x = \dfrac{\sin x}{\cos x}$ becomes infinite when $x \to \dfrac{\pi}{2}$

Consider $\displaystyle\int_0^M \tan x \, dx = \int_0^M \frac{\sin x}{\cos x} \, dx$ At the end, let $M \to \dfrac{\pi}{2}$

Sub $u = \cos x$ $\quad x = 0 \Rightarrow u = \cos 0 = 1$

$du = -\sin x \, dx$ $\quad x = M \Rightarrow u = \cos M$

Integral $= \displaystyle\int_{u=1}^{u=\cos M} \frac{1}{u} \cdot (-du) = \int_{\cos M}^{1} \frac{1}{u} \, du$

$= \Big[\ln|u| \Big]_{\cos M}^{1} = \ln|1| - \ln|\cos M|$

$= 0 - \ln|\cos M| = -\ln|\cos M|$

Now let $M \to \dfrac{\pi}{2}$

$\cos M \to 0$

So $\ln|\cos M| \to -\infty$

Limit does not exist

DIVERGES

12. Determine whether the integral converges or diverges, and find its value if it converges.

$$\int_0^4 \frac{1}{\sqrt{4-x}}\,dx \qquad x=4 \text{ is "problem"}$$

Consider $\displaystyle\int_0^M \frac{1}{\sqrt{4-x}}\,dx = \int_0^M (4-x)^{-1/2}\,dx$

At the end, let $M \to 4$

Sub $u = 4-x$ $\qquad x=0 \Rightarrow u = 4-0 = 4$

$du = -1\,dx$ $\qquad x = M \Rightarrow u = 4-M$

$-du = dx$

$$\int_{u=4}^{u=4-M} u^{-1/2}\cdot(-1)\,du = \int_{4-M}^{4} u^{-1/2}\,du$$

$$= \left[\frac{u^{1/2}}{1/2}\right]_{4-M}^{4} = 2\left[u^{1/2}\right]_{4-M}^{4}$$

$$= 2\left(4^{1/2} - (4-M)^{1/2}\right)$$

$$= 2\left(2 - \sqrt{4-M}\right)$$

If $M \to 4$, this approaches $2(2-0) = 4$

CONVERGES

13. Determine whether the integral converges or diverges, and find its value if it converges.

$$\int_0^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$$

Consider $\displaystyle\int_0^M \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$

Sub $u = -\sqrt{x} = -x^{1/2}$

$du = -\frac{1}{2}x^{-1/2}\,dx$

$-du = \frac{1}{2}\cdot\frac{1}{\sqrt{x}}\,dx$

$-2du = \frac{1}{\sqrt{x}}\,dx$

$x=0 \Rightarrow u = -\sqrt{0} = 0$

$x=M \Rightarrow u = -\sqrt{M}$

$$\int_{u=0}^{u=-\sqrt{M}} e^u\cdot(-2)\,du = 2\int_{-\sqrt{M}}^{0} e^u\,du$$

$$= 2\Big[e^u\Big]_{-\sqrt{M}}^{0} = 2\left(e^0 - e^{-\sqrt{M}}\right)$$

$$= 2\left(1 - e^{-\sqrt{M}}\right)$$

Now let $M\to\infty$

$\sqrt{M}\to\infty$

$-\sqrt{M}\to-\infty$

$e^{-\sqrt{M}}\to 0$

(or $\frac{1}{e^{\sqrt{M}}}$)

Answer: $2(1-0)$

$= 2$

CONVERGES