MAC2312
MORE suggested problems on Chapter 9 material (infinite series)

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1. Find the degree 2 Taylor polynomial for $f(x) = e^{-x}$ centered at $x_0 = 0$.

Taylor polynomial centered at $x_0 = 0$ will start

$$\underbrace{f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2}_{\text{Degree 2 Taylor polynomial}} + \cdots$$

$$\left.\begin{aligned} f(x) &= e^{-x} \\ f'(x) &= -e^{-x} \\ f''(x) &= e^{-x} \end{aligned}\right\} \Rightarrow \begin{aligned} f(0) &= 1 \\ f'(0) &= -1 \\ f''(0) &= 1 \end{aligned}$$

Answer: $1 + \frac{-1}{1!}x + \frac{1}{2!}x^2$

or $1 - x + \frac{x^2}{2}$

2. Find the degree 2 Taylor polynomial for $f(x) = \sqrt{x}$ centered at $x_0 = 1$, and use that to estimate $\sqrt{1.1}$.

$$f(x) = x^{1/2}$$

$$f'(x) = \frac{1}{2} x^{-1/2}$$

$$f''(x) = \frac{1}{2} \cdot \frac{-1}{2} x^{-3/2} = -\frac{1}{4} x^{-3/2}$$

Taylor polynomial centered at $x_0 = 1$ will start

$$\underbrace{f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2}_{\text{Degree 2}} + \ldots$$

$$f(1) = 1^{1/2} = 1$$

$$f'(1) = \frac{1}{2} \cdot 1^{-1/2} = \frac{1}{2}$$

$$f''(1) = -\frac{1}{4} \cdot 1^{-3/2} = \frac{-1}{4}$$

Degree 2 polynomial is

$$1 + \frac{1/2}{1!}(x-1) + \frac{-1/4}{2!}(x-1)^2$$

$$= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2$$

Estimate of $\sqrt{1.1} = f(1.1)$

is $1 + \frac{1}{2}(1.1-1) - \frac{1}{8}(1.1-1)^2$

$$= 1 + \frac{1}{2}(0.1) - \frac{1}{8}(0.1)^2 = 1 + \frac{0.1}{2} - \frac{0.01}{8} = 1 + 0.05 - 0.00125$$

$$= 1.04875$$

3. Find the MacLaurin series for the function $f(x) = e^{-x}$. Write your answer in sigma notation.

$$\left.\begin{aligned} f(x) &= e^{-x} \\ f'(x) &= -e^{-x} \\ f''(x) &= e^{-x} \\ f'''(x) &= -e^{-x} \\ &\text{etc.} \end{aligned}\right\} \Rightarrow \begin{aligned} f(0) &= e^0 = 1 \\ f'(0) &= -e^0 = -1 \\ f''(0) &= e^0 = 1 \\ f'''(0) &= -e^0 = -1 \end{aligned}$$

MacLaurin series is

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$$= 1 + \frac{-1}{1!}x + \frac{1}{2!}x^2 + \frac{-1}{3!}x^3 + \dots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k$$

4. Find the MacLaurin series for the function $f(x) = x \sin x$. Write your answer in sigma notation.

METHOD 1: $f(x) = x \sin x$

$$f'(x) = 1 \sin x + x \cos x$$

$$= \sin x + x \cos x$$

$$f''(x) = \cos x + 1 \cos x + x \cdot (-\sin x)$$

$$= 2\cos x - x \sin x$$

Derivatives may be complicated.

METHOD 2: We know the MacLaurin series for $\sin x$ is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

So, multiplying by x gives us MacLaurin series for $x \sin x$

$$x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{(2k+1)!}$$

5. Find the MacLaurin series for the function $f(x) = \ln(1+x)$. Write your answer in sigma notation.

$$\begin{aligned}
f(x) &= \ln(1+x) \\
f'(x) &= \frac{1}{1+x} = (1+x)^{-1} \\
f''(x) &= -1(1+x)^{-2} \\
f'''(x) &= +2(1+x)^{-3} \\
f^{(4)}(x) &= -3\cdot 2(1+x)^{-4} \\
f^{(5)}(x) &= +4\cdot 3\cdot 2(1+x)^{-5} \\
&\text{etc.}
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
f(0) &= \ln 1 = 0 \\
f'(0) &= 1 \\
f''(0) &= -1 \\
f'''(0) &= +2! \\
f^{(4)}(0) &= -3! \\
f^{(5)}(0) &= +4! \\
&\text{etc.}
\end{aligned}$$

So $f^{(k)}(0) = (-1)^{k-1}(k-1)!$ if $k \geq 1$

MacLaurin series is

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$$= 0 + \frac{1}{1!}x + \frac{-1}{2!}x^2 + \frac{2!}{3!}x^3 + \frac{-3!}{4!}x^4 + \dots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}x^k}{k}$$

Notice the denominators are not factorials.
So maybe this series doesn't converge as quickly...

6. Find the MacLaurin series for the function $f(x) = \sqrt{1+x}$. Write your answer in sigma notation.

$$f(x) = (1+x)^{1/2}$$

$$f'(x) = \frac{1}{2}(1+x)^{-1/2}$$

$$f''(x) = \frac{1}{2} \cdot \frac{-1}{2}(1+x)^{-3/2}$$

$$f'''(x) = \frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2}(1+x)^{-5/2}$$

$$f^{(4)}(x) = \frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2} \cdot \frac{-5}{2}(1+x)^{-7/2}$$

So then

$$\left.\begin{aligned} f(0) &= 1 \\ f'(0) &= \frac{1}{2} \\ f''(0) &= -\frac{1}{2^2} \\ f'''(0) &= \frac{+1\cdot 3}{2^3} \\ f^{(4)}(0) &= \frac{-1\cdot 3\cdot 5}{2^4} \end{aligned}\right\}$$

How to write this in general ?

$$f^{(k)}(0) = \frac{(-1)^{k-1}\, 1\cdot 3\cdot 5 \cdots (2k-3)}{2^k}$$

if $k \geq 1$

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So MacLaurin series is

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$

$$= 1 + \frac{\frac{1}{2}}{1!}x + \frac{\frac{-1}{2^2}}{2!}x^2 + \frac{\frac{+1\cdot 3}{2^3}}{3!}x^3 + \frac{\frac{-1\cdot 3\cdot 5}{2^4}}{4!}x^4 + \dots$$

$$= 1 + \frac{1}{2\cdot 1!}x - \frac{1}{2^2\cdot 2!}x^2 + \frac{1\cdot 3}{2^3\cdot 3!}x^3 - \frac{1\cdot 3\cdot 5}{2^4\cdot 4!}x^4 + \dots$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\, 1\cdot 3\cdot 5\cdots(2k-3)}{2^k\cdot k!}x^k$$

7. Find the Taylor series centered at $x_0 = 1$ for the function $f(x) = e^x$. Write your answer in sigma notation.

$$\left.\begin{aligned} f(x) &= e^x \\ f'(x) &= e^x \\ f''(x) &= e^x \\ &\text{etc.} \end{aligned}\right\} \Rightarrow \begin{aligned} f(1) &= e \\ f'(1) &= e \\ f''(1) &= e \\ &\text{etc.} \end{aligned}$$

Taylor series centered at $x_0 = 1$ is

$$f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \dots$$

$$= e + \frac{e}{1!}(x-1) + \frac{e}{2!}(x-1)^2 + \frac{e}{3!}(x-1)^3 + \dots$$

$$= \sum_{k=0}^{\infty} \frac{e(x-1)^k}{k!}$$

By the way, this could be used to estimate something like $e^{1.1}$.

$$e^{1.1} \approx e + \frac{e}{1!}(0.1) + \frac{e}{2!}(0.1)^2 + \dots$$

8. Find the interval of convergence of the power series, and find a familiar function that is represented by the power series on that interval.

$$1 - x + x^2 - x^3 + \cdots + (-1)^k x^k + \cdots$$

$$\sum_{k=0}^{\infty} (-1)^k x^k = \sum_{k=0}^{\infty} (-x)^k$$

This is a geometric series with $r = -x$

So it converges if $-x$ is between -1 and 1

i.e. if x is between -1 and 1.

If it converges, the sum is $\dfrac{a}{1-r}$

where $a =$ first term $= 1$

Sum of series (if $-1 < x < 1$) is $\dfrac{1}{1-(-x)}$

$$= \frac{1}{1+x}$$

9. Find the radius of convergence and the interval of convergence.

$$\sum_{k=0}^{\infty} \frac{x^k}{k+1}$$

$$a_k = \frac{x^k}{k+1} \qquad a_{k+1} = \frac{x^{k+1}}{k+2}$$

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{x^{k+1}}{k+2} \div \frac{x^k}{k+1} \right|$$

$$= \lim_{k\to\infty} \left| \frac{x^{k+1}}{k+2} \cdot \frac{k+1}{x^k} \right| = \lim_{k\to\infty} \left| \frac{x^{k+1}}{x^k} \cdot \frac{k+1}{k+2} \right|$$

$$= \lim_{k\to\infty} \left(|x| \cdot \underbrace{\frac{k+1}{k+2}}_{\to 1} \right) = |x|$$

So the series converges absolutely if $|x| < 1$

i.e. $-1 < x < 1$.

Radius of convergence is 1.

Endpoints? $x = -1$? $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}$ Convergent alternating series.

$x = 1$? $\displaystyle\sum_{k=0}^{\infty} \frac{1^k}{k+1}$ Divergent harmonic series.

Interval of convergence is $[-1, 1)$

10. Find the radius of convergence and the interval of convergence.

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!}$$

$$a_k = \frac{(-1)^k x^k}{k!} \qquad a_{k+1} = \frac{(-1)^{k+1} x^{k+1}}{(k+1)!}$$

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{(-1)^{k+1} x^{k+1}}{(k+1)!} \div \frac{(-1)^k x^k}{k!} \right|$$

$$= \lim_{k\to\infty} \left| \frac{(-1)^{k+1} x^{k+1}}{(k+1)!} \cdot \frac{k!}{(-1)^k x^k} \right| = \lim_{k\to\infty} \left| \frac{x^{k+1}}{x^k} \cdot \frac{k!}{(k+1)!} \right|$$

$$= \lim_{k\to\infty} \left| x \cdot \frac{1}{k+1} \right| = \lim_{k\to\infty} \left(|x| \cdot \underbrace{\frac{1}{k+1}}_{\to 0} \right) = 0$$

So $\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = 0 < 1$ <u>regardless</u> of the value of x

So $\sum a_k$ converges absolutely for <u>all</u> values of x

Radius of convergence and interval of convergence are infinite.