MAC2312
Suggested problems on Chapter 9 material
(infinite series)

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1. Find a formula for the general term of the sequence.

$$1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$$

Can use $a_n = \dfrac{1}{3^n}$ if starting at $n = 0$.

Can also use $a_n = \dfrac{1}{3^{n-1}}$ if starting at $n = 1$.

2. Find a formula for the general term of the sequence.

$$1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \ldots$$

Can use $a_n = \dfrac{(-1)^n}{3^n}$ or $\left(-\dfrac{1}{3}\right)^n$ starting at $n=0$

Can also use $a_n = \dfrac{(-1)^{n-1}}{3^{n-1}}$ or $\dfrac{(-1)^{n+1}}{3^{n-1}}$ starting at $n=1$

3. Find a formula for the general term of the sequence.

$$\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \ldots$$

Can use $a_n = \dfrac{2n-1}{2n}$ if starting at $n=1$

Can also be written $1 - \dfrac{1}{2n}$

Other possible correct answers include

$a_n = \dfrac{2n+1}{2n+2}$ if starting at $n=0$

4. Find a formula for the general term of the sequence.

$$\frac{1}{\pi^{1/2}}, \frac{4}{\pi^{1/3}}, \frac{9}{\pi^{1/4}}, \frac{16}{\pi^{1/5}}, \ldots$$

$a_n = \dfrac{n^2}{\pi^{1/(n+1)}}$ if starting at $n=1$

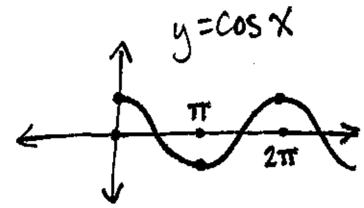
or $a_n = \dfrac{(n-1)^2}{\pi^{1/n}}$ if starting at $n=2$

or $a_n = \dfrac{(n+1)^2}{\pi^{1/(n+2)}}$ starting at $n=0$

5. (a) Write out the first four terms of the sequence $\{1+(-1)^n\}$, starting with $n=0$.
(b) Write out the first four terms of the sequence $\{\cos n\pi\}$, starting with $n=0$.
(c) Use the results in parts (a) and (b) to express the general term of the sequence $\{4, 0, 4, 0, \ldots\}$ in two different ways, starting with $n=0$.

(a) $2, 0, 2, 0$ because
$$1+(-1)^0 = 1+1 = 2$$
$$1+(-1)^1 = 1-1 = 0$$
$$1+(-1)^2 = 1+1 = 2$$
$$1+(-1)^3 = 1-1 = 0$$

(b) $1, -1, 1, -1$ because
$$\cos 0 = 1$$
$$\cos \pi = -1$$
$$\cos 2\pi = 1$$
$$\cos 3\pi = -1$$



(c) One way to get $4, 0, 4, 0$ is 2 times sequence (a)

so $2\left(1+(-1)^n\right)$

Another way might be to use (b) somehow.
Notice $1+$(sequence b) will give $2, 0, 2, 0$. Can then multiply by 2

So another answer is $2(1+\cos n\pi)$

6. Determine whether the sequence converges. If it converges, find its limit. Also write out the first five terms of the sequence.

$$\left\{\frac{n}{n+2}\right\}_{n=1}^{\infty}$$

$$\lim_{n\to\infty} \frac{n}{n+2} = \lim_{n\to\infty} \frac{n}{n+2} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{n}{n}+\frac{2}{n}} = \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}}$$

$$= \frac{1}{1+0} = \frac{1}{1} = 1.$$

Since $\lim_{n\to\infty} \frac{n}{n+2}$ exists (and is a finite number), the sequence CONVERGES.

The first five terms are $\frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6}, \frac{5}{7}$

(NOTE: Any similar-looking question about SERIES, i.e. infinite SUMS would be a totally different question!)

7. Determine whether the sequence converges. If it converges, find its limit. Also write out the first five terms of the sequence.

$$\left\{n \sin \frac{\pi}{n}\right\}_{n=1}^{\infty}$$

When n gets big, $\frac{\pi}{n}$ gets small.

FACT: $\sin x \approx x$ if x is near 0.

More specifically, $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

Now $\lim_{n \to \infty} n \sin \frac{\pi}{n} = \lim_{n \to \infty} \frac{\sin \frac{\pi}{n}}{\frac{1}{n}} \cdot \frac{\pi}{\pi}$

$= \lim_{n \to \infty} \pi \cdot \underbrace{\frac{\sin \frac{\pi}{n}}{\frac{\pi}{n}}}_{\to 1} = \pi \cdot 1 = \pi$. Sequence CONVERGES.

because if $x = \frac{\pi}{n}$ then $x \to 0$

and $\frac{\sin \frac{\pi}{n}}{\frac{\pi}{n}}$ has the form $\frac{\sin x}{x}$

First five terms: $\sin \pi,\ 2 \sin \frac{\pi}{2},\ 3 \sin \frac{\pi}{3},\ 4 \sin \frac{\pi}{4},\ 5 \sin \frac{\pi}{5}$

or $0,\ 2,\ 3 \cdot \frac{\sqrt{3}}{2},\ 4 \cdot \frac{\sqrt{2}}{2},\ 5 \sin \frac{\pi}{5}$

8. Determine whether the sequence converges. If it converges, find its limit. Also write out the first five terms of the sequence.

$$\{1 + (-1)^n\}_{n=1}^{\infty}$$

$\lim_{n\to\infty} \left(1 + (-1)^n \right)$ does not exist.

Sequence does not converge.

Terms of sequence are $0, 2, 0, 2, 0, 2, \ldots$

$1+(-1)^1 = 1-1 = 0$

$1+(-1)^2 = 1+1 = 2$

$1+(-1)^3 = 1-1 = 0$

$1+(-1)^4 = 1+1 = 2$

etc.

Terms of sequence do <u>not</u> get arbitrarily close to one <u>particular</u> number.

9. Find an expression for the general term of the sequence. Determine whether the sequence converges. If it converges, find its limit.

$$\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \ldots$$

Can use $a_n = \dfrac{2n-1}{2n}$ if starting at $n=1$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n-1}{2n} = \lim_{n\to\infty} \frac{2n-1}{2n} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{\frac{2n}{n} - \frac{1}{n}}{\frac{2n}{n}} = \lim_{n\to\infty} \frac{2 - \frac{1}{n}}{2} = \frac{2-0}{2}$$

$= \dfrac{2}{2} = 1.$ The sequence converges to 1.

(Again, we are NOT talking about SERIES right now.)

10. Find an expression for the general term of the sequence. Determine whether the sequence converges. If it converges, find its limit.

$$\frac{1}{3}, -\frac{1}{9}, \frac{1}{27}, -\frac{1}{81}, \dots$$

Can use $a_n = \dfrac{(-1)^{n-1}}{3^n}$ or $\dfrac{(-1)^{n+1}}{3^n}$ starting at $n=1$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^{n+1}}{3^n} = \lim_{n\to\infty} \frac{(-1)(-1)^n}{3^n}$$

$$= (-1)\lim_{n\to\infty}\left(\frac{-1}{3}\right)^n = (-1)\cdot 0 = 0$$

So sequence converges

Here we have used the rule that $\lim_{n\to\infty} r^n = 0$ if r is a constant between -1 and 1.

11. Determine whether the sequence is strictly increasing, strictly decreasing, or neither.

$$\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$$

Sequence is $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

For purposes of this course, we can say it's "obvious" that n is strictly <u>in</u>creasing and $\frac{1}{n}$ is strictly <u>de</u>creasing.

12. Determine whether the sequence is strictly increasing, strictly decreasing, or neither.

$$\left\{\frac{n}{2n+1}\right\}_{n=1}^{\infty}$$

To decide, can compare n^{th} term to $(n+1)^{th}$ term.

$$a_n = \frac{n}{2n+1}. \quad a_{n+1} = \frac{n+1}{2(n+1)+1} = \frac{n+1}{2n+3}$$

Do we have $a_n < a_{n+1}$ or $a_n > a_{n+1}$ or neither?

$$\frac{n}{2n+1} \overset{?}{\lessgtr} \frac{n+1}{2n+3}$$

$\downarrow$ multiply both sides by $(2n+1)(2n+3)$ (NOTE: This step is reversible)

$$n(2n+3) \overset{?}{\lessgtr} (n+1)(2n+1)$$

$$2n^2+3n \overset{?}{\lessgtr} 2n^2+3n+1$$ Right side is bigger

$$2n^2+3n < 2n^2+3n+1 \Rightarrow n(2n+3) < (n+1)(2n+1)$$

$$\Rightarrow \frac{n}{2n+1} < \frac{n+1}{2n+3}$$ i.e. $a_n < a_{n+1}$

The sequence is STRICTLY INCREASING

13. Determine whether the sequence is strictly increasing, strictly decreasing, or neither.

$$\left\{\frac{10^n}{(2n)!}\right\}_{n=1}^{\infty}$$

$$a_n = \frac{10^n}{(2n)!} \qquad a_{n+1} = \frac{10^{n+1}}{(2(n+1))!} = \frac{10^{n+1}}{(2n+2)!}$$

To decide if $a_n < a_{n+1}$ or $a_n > a_{n+1}$,
could look at ratio:

$$\frac{a_{n+1}}{a_n} = \frac{10^{n+1}}{(2n+2)!} \div \frac{10^n}{(2n)!} = \frac{10^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{10^n}$$

$$= \frac{10^{n+1}}{10^n} \cdot \frac{(2n)!}{(2n+2)!} = 10 \cdot \frac{1}{(2n+1)(2n+2)}$$

We're only using $n \geq 1$, so $(2n+1)(2n+2) \geq (2+1)(2+2)$
$= 3 \cdot 4 = 12$. So $\frac{a_{n+1}}{a_n} = \frac{10}{(2n+1)(2n+2)} \leq \frac{10}{12} < 1$

so $a_{n+1} < a_n$ always. So the sequence is STRICTLY DECREASING.

14. Determine whether the series converges. If possible, find its sum.

$$\sum_{k=1}^{\infty} \left(-\frac{3}{4}\right)^{k-1}$$

This is a geometric series with $r = \frac{-3}{4}$.

Series is $\left(-\frac{3}{4}\right)^0 + \left(-\frac{3}{4}\right)^1 + \left(-\frac{3}{4}\right)^2 + \dots$

Since the series is geometric with $-1 < r < 1$, the series CONVERGES.

The value of the sum is $\frac{a}{1-r}$ where a is the first term.

$a = \left(-\frac{3}{4}\right)^0 = 1$

Sum of series is $\frac{a}{1-r} = \frac{1}{1-\left(-\frac{3}{4}\right)} = \frac{1}{1+\frac{3}{4}}$

$= \frac{1}{\frac{7}{4}} = \frac{4}{7}$.

15. Determine whether the series converges. If possible, find its sum.

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{7}{6^{k-1}}$$

This is a geometric series with $r = -\frac{1}{6}$.

The k^{th} term is $(-1)^{k-1}\cdot\frac{7}{6^{k-1}} = 7\cdot\left(-\frac{1}{6}\right)^{k-1}$.

The series starts: $7\cdot\left(-\frac{1}{6}\right)^0 + 7\cdot\left(-\frac{1}{6}\right)^1 + 7\cdot\left(-\frac{1}{6}\right)^2 + \cdots$

Since the series is geometric with $-1 < r < 1$, the series CONVERGES.

The value of the sum is $\frac{a}{1-r}$ where a is the first term.

$a = 7\cdot\left(-\frac{1}{6}\right)^0 = 7$

Sum of series is $\frac{a}{1-r} = \frac{7}{1-\left(-\frac{1}{6}\right)} = \frac{7}{1+\frac{1}{6}}$

$= \frac{7}{\frac{7}{6}} = 7\cdot\frac{6}{7} = 6.$

16. Determine whether the series converges. If possible, find its sum.

$$\sum_{k=1}^{\infty} \frac{1}{(k+2)(k+3)}$$

The usual way to determine if this series converges is to notice the k^{th} term is "similar" to $\frac{1}{k^2}$ and use one of the comparison tests.

For most series, we <u>cannot</u> find the sum, but for this series, we can use a "trick".

k^{th} term is $\dfrac{1}{(k+2)(k+3)} = \dfrac{(k+3)-(k+2)}{(k+2)(k+3)} = \dfrac{k+3}{(k+2)(k+3)} - \dfrac{k+2}{(k+2)(k+3)}$

$= \dfrac{1}{k+2} - \dfrac{1}{k+3}$. Then n^{th} partial sum is

$$S_n = \sum_{k=1}^{n} \frac{1}{(k+2)(k+3)} = \sum_{k=1}^{n} \left(\frac{1}{k+2} - \frac{1}{k+3} \right)$$

$$= \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \dots + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \left(\frac{1}{n+2} - \frac{1}{n+3}\right)$$

$= \dfrac{1}{3} - \dfrac{1}{n+3}$ (telescoping sum)

Then sum of <u>infinite</u> series is $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{3} - \frac{1}{n+3}\right)$

$= \dfrac{1}{3}$.

17. Determine whether the series converges. If possible, find its sum.

$$\sum_{k=2}^{\infty} \frac{1}{k^2-1}$$

This is very similar to the previous question.

Trick: k^{th} term is $\frac{1}{k^2-1} = \frac{1}{(k-1)(k+1)} = \frac{1}{2} \cdot \frac{2}{(k-1)(k+1)}$

$= \frac{1}{2} \cdot \frac{(k+1)-(k-1)}{(k-1)(k+1)} = \frac{1}{2}\left(\frac{k+1}{(k-1)(k+1)} - \frac{k-1}{(k-1)(k+1)}\right)$

$= \frac{1}{2}\left(\frac{1}{k-1} - \frac{1}{k+1}\right)$. Then n^{th} partial sum is

$$S_n = \sum_{k=2}^{n} \frac{1}{k^2-1} = \sum_{k=2}^{n} \frac{1}{2}\left(\frac{1}{k-1} - \frac{1}{k+1}\right) = \frac{1}{2}\sum_{k=2}^{n}\left(\frac{1}{k-1} - \frac{1}{k+1}\right)$$

$$= \frac{1}{2}\left[\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \dots + \left(\frac{1}{n-2} - \frac{1}{n}\right) + \left(\frac{1}{n-1} - \frac{1}{n+1}\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-2} + \frac{1}{n-1}\right) - \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots + \frac{1}{n} + \frac{1}{n+1}\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{1}{1} + \frac{1}{2}\right) - \left(\frac{1}{n} + \frac{1}{n+1}\right)\right]$$

Then sum of <u>infinite</u> series is

$$\lim_{n\to\infty} S_n = \frac{1}{2}\left(1 + \frac{1}{2} - 0\right) = \frac{3}{4}$$

18. Determine whether the series converges or diverges.

$$\sum_{k=1}^{\infty} k^{-4/3}$$

$$\sum_{k=1}^{\infty} k^{-4/3} = \sum_{k=1}^{\infty} \frac{1}{k^{4/3}}$$

This is a p-series with $p = \frac{4}{3}$.

Since $p = \frac{4}{3} > 1$, this is a CONVERGENT p-series.

19. Determine whether the series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{1}{k^{1/4}}$$

This is a p-series with $p = \frac{1}{4}$.

Since $p = \frac{1}{4} \leq 1$,

this is a DIVERGENT p-series.

20. Determine whether the series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{1}{k^{\pi}}$$

This is a p-series with $p = \pi \approx 3.14$.

Since $p = \pi > 1$,

this is a CONVERGENT p-series.

21. Determine whether the series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{k^2+k+3}{2k^2+1}$$

Notice the k^{th} term of the series is "like" $\frac{k^2}{2k^2}$.

More specifically, notice $\lim_{k\to\infty} \frac{k^2+k+3}{2k^2+1}$

$$= \lim_{k\to\infty} \frac{k^2+k+3}{2k^2+1} \cdot \frac{\frac{1}{k^2}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{\frac{k^2}{k^2}+\frac{k}{k^2}+\frac{3}{k^2}}{\frac{2k^2}{k^2}+\frac{1}{k^2}}$$

$$= \lim_{k\to\infty} \frac{1+\frac{1}{k}+\frac{3}{k^2}}{2+\frac{1}{k^2}} = \frac{1+0+0}{2+0} = \frac{1}{2}$$

Since the k^{th} term of the series does NOT approach 0, the series MUST diverge! ("Divergence pre-test")

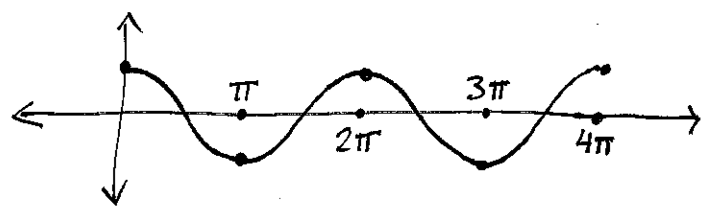
Rough idea: Since the terms of the series are not even getting small, the series has NO CHANCE of converging.

22. Determine whether the series converges or diverges.

$$\sum_{k=1}^{\infty} \cos k\pi$$

Note: $\cos k\pi$ is either 1 or -1 depending on whether k is even or odd.

$\cos \pi = -1$
$\cos 2\pi = 1$
$\cos 3\pi = -1$
$\cos 4\pi = 1$
etc.



So k^{th} term of series is always 1 or -1

So k^{th} term of series does NOT approach 0

So series DIVERGES by "Divergence pre-test".

We could also consider partial sums. Series is

$\cos \pi + \cos 2\pi + \cos 3\pi + \cos 4\pi + \ldots$

which is $-1 + 1 - 1 + 1 - 1 + 1 - \ldots$

So partial sums are:

$-1 = -1$
$-1 + 1 = 0$
$-1 + 1 - 1 = -1$
$-1+1-1+1 = 0$
$-1+1-1+1-1 = -1$
$-1+1-1+1-1+1 = 0$
etc.

23. Determine whether the series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{1}{5k+2}$$

Notice k^{th} term is "similar" to $\frac{1}{5k} = \frac{1}{5} \cdot \frac{1}{k}$

So series is "similar" to $\sum \frac{1}{5k} = \frac{1}{5} \sum \frac{1}{k}$

METHOD 1: Limit comparison test.

k^{th} term of given series is $a_k = \frac{1}{5k+2}$. Try $b_k = \frac{1}{k}$.

Then $\lim\limits_{k\to\infty} \frac{a_k}{b_k} = \lim\limits_{k\to\infty} \left(\frac{1}{5k+2} \div \frac{1}{k} \right) = \lim\limits_{k\to\infty} \left(\frac{1}{5k+2} \cdot \frac{k}{1} \right)$

$= \lim\limits_{k\to\infty} \frac{k}{5k+2} \cdot \frac{\frac{1}{k}}{\frac{1}{k}} = \frac{1}{5}$ which is NOT 0, NOT ∞.

So $\sum a_k$ "behaves the same" as $\sum b_k = \sum \frac{1}{k}$ which DIVERGES.

METHOD 2: Could use basic comparison, but need to be clever.

Notice $5k+2 \leq 5k+2k = 7k$, so $\frac{1}{5k+2} \geq \frac{1}{7k}$

So $\sum \frac{1}{5k+2} \geq \sum \frac{1}{7k} = \frac{1}{7} \underbrace{\sum \frac{1}{k}}_{\text{DIVERGES}}$

METHOD 3: INTEGRAL TEST!

which forces the series $\sum \frac{1}{5k+2}$ to diverge. $\int_1^{\infty} \frac{1}{5x+2}\,dx = ?$

24. Determine whether the series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{1}{k+6}$$

This series is

$$\frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \dots$$

which is the same as the harmonic series except the first few terms have been removed.

So the series DIVERGES.

Other possible methods:

(i) Limit comparison test with $a_k = \frac{1}{k+6}$ and $b_k = \frac{1}{k}$

(ii) Basic comparison test. Can say e.g. $k+6 \leq k+6k = 7k$

So $\frac{1}{k+6} \geq \frac{1}{7k}$ so $\sum_{k=1}^{\infty} \frac{1}{k+6} \geq \sum_{k=1}^{\infty} \frac{1}{7k} = \frac{1}{7} \sum_{k=1}^{\infty} \frac{1}{k}$

(iii) Integral test. $\int_1^{\infty} \frac{1}{x+6}\, dx = ?$ Sub $u = x+6$

$\int_1^M \frac{1}{x+6}\, dx = \Big[\ln|x+6|\Big]_1^M = \dots$ Diverges because $\ln \infty = \infty$

25. Determine whether the series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{k}{\ln(k+1)}$$

First, what do we guess happens?

k^{th} term is $\dfrac{k}{\ln(k+1)}$. Top grows linearly. Bottom grows slowly.

We suspect the top grows faster than the bottom which would mean the terms don't even approach 0 So the series would have no chance of converging. (Divergence pre-test)

$$\lim_{k\to\infty} \frac{k}{\ln(k+1)} = ?$$ Has form $\frac{\infty}{\infty}$, so use L'Hopital.

$$\lim_{k\to\infty} \frac{1}{\frac{1}{k+1}} = \lim_{k\to\infty} (k+1) = \infty, \text{ not } 0$$

So as we suspected, k^{th} term does NOT approach 0.

Series DIVERGES.

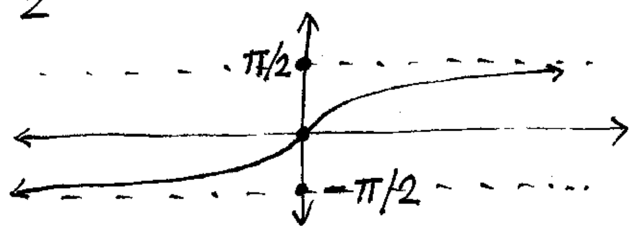
26. Determine whether the series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{\arctan k}{1+k^2}$$

We need to know something about the arctangent function.

Fact: $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$, and $0 < \arctan x < \frac{\pi}{2}$ if x is positive

Graph of $y = \arctan x$:



So $0 < \arctan k < \pi/2 \Rightarrow 0 < \dfrac{\arctan k}{1+k^2} < \dfrac{\pi/2}{1+k^2}$

Then $\dfrac{\pi/2}{1+k^2} < \dfrac{\pi/2}{k^2}$ so $\sum \dfrac{\arctan k}{1+k^2} < \sum \dfrac{\pi/2}{1+k^2} < \sum \dfrac{\pi/2}{k^2}$

$= \dfrac{\pi}{2} \sum \dfrac{1}{k^2}$ which is a constant times a CONVERGENT p-series.

By Basic comparison, $\sum \dfrac{\arctan k}{1+k^2}$ converges.

ANOTHER METHOD: Integral test. $\displaystyle\int_1^{\infty} \frac{\arctan x}{1+x^2}\,dx$

Consider $\displaystyle\int_1^{M} \frac{\arctan x}{1+x^2}\,dx$. The substitution $u = \arctan x$ will work, because $du = \dfrac{1}{1+x^2}\,dx$

27. Determine whether the series converges or diverges.

$$\sum_{k=5}^{\infty} 7k^{-1.01}$$

$$\sum_{k=5}^{\infty} 7 \cdot \frac{1}{k^{1.01}} = 7 \underbrace{\sum_{k=5}^{\infty} \frac{1}{k^{1.01}}}_{\substack{p\text{-series with } p = 1.01 > 1 \\ \text{CONVERGES}}}$$

Starting at $k=5$ instead of $k=1$ only gets rid of a finite number of terms and does not change the "big picture" question of whether the series converges or diverges.

Answer: Series CONVERGES

28. Determine whether the series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{1}{5k^2 - k}$$

k^{th} term of series $a_k = \dfrac{1}{5k^2-k}$ "resembles" $\dfrac{1}{5k^2}$

Try Limit Comparison with $b_k = \dfrac{1}{k^2}$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty}\left(\frac{1}{5k^2-k} \div \frac{1}{k^2}\right) = \lim_{k\to\infty}\left(\frac{1}{5k^2-k}\cdot\frac{k^2}{1}\right)$$

$$= \lim_{k\to\infty} \frac{k^2}{5k^2-k}\cdot\frac{\frac{1}{k^2}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{\frac{k^2}{k^2}}{\frac{5k^2}{k^2} - \frac{k}{k^2}}$$

$$= \lim_{k\to\infty} \frac{1}{5 - \frac{1}{k}} = \frac{1}{5-0} = \frac{1}{5}$$ NOT 0, NOT ∞

Since $\displaystyle\lim_{k\to\infty} \frac{a_k}{b_k} = \frac{1}{5}$ is not 0 and not ∞,

we conclude $\sum a_k = \sum \dfrac{1}{5k^2-k}$ "behaves the same"

as $\sum b_k = \sum \dfrac{1}{k^2}$ which CONVERGES (p-series with $p = 2 > 1$)

29. Determine whether the series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{3}{k - \frac{1}{4}}$$

$a_k = \dfrac{3}{k - \frac{1}{4}}$. Try Limit Comparison with $b_k = \dfrac{1}{k}$.

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty}\left(\frac{3}{k-\frac{1}{4}} \div \frac{1}{k}\right) = \lim_{k\to\infty}\left(\frac{3}{k-\frac{1}{4}}\cdot\frac{k}{1}\right)$$

$$= \lim_{k\to\infty} \frac{3k}{k-\frac{1}{4}} \cdot \frac{\frac{1}{k}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{3}{1-\frac{1}{4k}} = \frac{3}{1-0} = 3$$

Since $\lim_{k\to\infty} \frac{a_k}{b_k} = 3$ is not 0 and not ∞, we conclude that $\sum a_k = \sum \frac{3}{k-\frac{1}{4}}$ "behaves the same" as $\sum b_k = \sum \frac{1}{k}$ which DIVERGES. (Harmonic series. Also p-series with $p=1\leq 1$.)

Another method: Could use basic comparison.

$k - \frac{1}{4} < k$ so $\dfrac{1}{k-\frac{1}{4}} > \dfrac{1}{k}$, and then $\sum \dfrac{3}{k-\frac{1}{4}}$

is greater than $\sum \frac{3}{k} = 3\sum\frac{1}{k}$ which DIVERGES

30. Determine whether the series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{1}{3^k + 5}$$

k^{th} term $a_k = \dfrac{1}{3^k+5}$ "resembles" $\dfrac{1}{3^k}$

Try Limit Comparison with $b_k = \dfrac{1}{3^k}$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty}\left(\frac{1}{3^k+5} \div \frac{1}{3^k}\right) = \lim_{k\to\infty}\left(\frac{1}{3^k+5}\cdot\frac{3^k}{1}\right)$$

$$= \lim_{k\to\infty} \frac{3^k}{3^k+5}\cdot\frac{\frac{1}{3^k}}{\frac{1}{3^k}} = \lim_{k\to\infty} \frac{1}{1+\frac{5}{3^k}}$$

$$= \frac{1}{1+0} = 1$$ Not 0, Not ∞.

Since $\displaystyle\lim_{k\to\infty}\frac{a_k}{b_k} = 1$ is not 0 and not ∞

we conclude that $\sum a_k = \sum \frac{1}{3^k+5}$ "behaves the same"

as $\sum b_k = \sum \frac{1}{3^k}$ which CONVERGES (geometric with $r = \frac{1}{3}$, $-1 < r < 1$)

31. Determine whether the series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{\ln k}{k}$$

$$\frac{\ln 1}{1} + \frac{\ln 2}{2} + \frac{\ln 3}{3} + \cdots$$

Note: If $k \geq 3$ then $\ln k \geq \ln 3 > \ln e = 1$

So then $\frac{\ln k}{k} > \frac{1}{k}$ so $\sum_{k=3}^{\infty} \frac{\ln k}{k} > \sum_{k=3}^{\infty} \frac{1}{k}$

So $\sum_{k=3}^{\infty} \frac{\ln k}{k}$ diverges by Basic Comparison.

Then the given series $\sum_{k=1}^{\infty} \frac{\ln k}{k}$ also diverges (just two more terms).

ANOTHER METHOD: Integral test will work.

$\int_1^{\infty} \frac{\ln x}{x}\,dx = ?$ Consider $\int_1^{M} \frac{\ln x}{x}\,dx$

Sub $u = \ln x$
$\Rightarrow du = \frac{1}{x}\,dx$

Integral $= \int_{u=0}^{u=\ln M} u\,du = \left[\frac{u^2}{2}\right]_0^{\ln M}$

$= \frac{(\ln M)^2}{2}$

which approaches ∞ when $M \to \infty$

32. Determine whether the series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{4k^2 - 2k + 6}{8k^7 + k - 8}$$

k^{th} term is $a_k = \dfrac{4k^2 - 2k + 6}{8k^7 + k - 8}$ Try Limit Comparison with $b_k = \dfrac{k^2}{k^7} = \dfrac{1}{k^5}$

$$\lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \left(\frac{4k^2 - 2k + 6}{8k^7 + k - 8} \div \frac{1}{k^5} \right)$$

$$= \lim_{k \to \infty} \left(\frac{4k^2 - 2k + 6}{8k^7 + k - 8} \cdot \frac{k^5}{1} \right) = \lim_{k \to \infty} \frac{4k^7 - 2k^6 + 6k^5}{8k^7 + k - 8}$$

$$= \lim_{k \to \infty} \frac{4k^7 + (\text{smaller powers})}{8k^7 + (\text{smaller powers})} = \lim_{k \to \infty} \frac{4k^7}{8k^7} = \frac{1}{2}$$ Not 0 Not ∞

Since $\displaystyle\lim_{k \to \infty} \frac{a_k}{b_k} = \frac{1}{2}$ is not 0 and not ∞,

we conclude that $\sum a_k$ "behaves the same" as $\sum b_k = \sum \frac{1}{k^5}$

which converges (p-series with $p = 5 > 1$) so $\sum a_k$ CONVERGES.

33. Determine whether the series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{1}{9k+6}$$

$a_k = \dfrac{1}{9k+6}$. Can do Limit Comparison with $b_k = \dfrac{1}{k}$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty}\left(\frac{1}{9k+6} \div \frac{1}{k}\right) = \lim_{k\to\infty}\left(\frac{1}{9k+6}\cdot\frac{k}{1}\right)$$

$$= \lim_{k\to\infty} \frac{k}{9k+6}\cdot\frac{\frac{1}{k}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{1}{9+\frac{6}{k}} = \frac{1}{9}$$

Since $\displaystyle\lim_{k\to\infty} \frac{a_k}{b_k} = \frac{1}{9}$ is not 0 and not ∞,

we conclude that $\sum a_k = \sum \dfrac{1}{9k+6}$ "behaves the same"

as $\sum b_k = \sum \dfrac{1}{k}$ which diverges (harmonic series)

So $\sum a_k = \sum \dfrac{1}{9k+6}$ DIVERGES.

34. Determine whether the series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{3^k}{k!}$$

Ratio test will probably work (often works with factorials and exponential functions)

$$a_k = \frac{3^k}{k!} \qquad a_{k+1} = \frac{3^{k+1}}{(k+1)!}$$

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty}\left(\frac{3^{k+1}}{(k+1)!} \div \frac{3^k}{k!}\right) = \lim_{k\to\infty}\left(\frac{3^{k+1}}{(k+1)!} \cdot \frac{k!}{3^k}\right)$$

$$= \lim_{k\to\infty}\left(\frac{3^{k+1}}{3^k} \cdot \frac{k!}{(k+1)!}\right) = \lim_{k\to\infty}\left(3 \cdot \frac{1}{k+1}\right)$$

$$= \lim_{k\to\infty} \frac{3}{k+1} = 0 < 1$$

$$\frac{3}{\infty}$$

Since $\lim\limits_{k\to\infty} \frac{a_{k+1}}{a_k} < 1$, the ratio test says that the series $\sum a_k = \sum \frac{3^k}{k!}$ CONVERGES.

35. Determine whether the series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{k}{k^2+1}$$

k^{th} term is $a_k = \dfrac{k}{k^2+1}$. Try Limit Comparison with $b_k = \dfrac{1}{k}$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty}\left(\frac{k}{k^2+1} \div \frac{1}{k}\right) = \lim_{k\to\infty}\left(\frac{k}{k^2+1}\cdot\frac{k}{1}\right)$$

$$= \lim_{k\to\infty} \frac{k^2}{k^2+1} = 1 \qquad \frac{k^2}{k^2+1}\cdot\frac{\frac{1}{k^2}}{\frac{1}{k^2}} = \frac{1}{1+\frac{1}{k^2}}$$

Since $\lim\limits_{k\to\infty} \dfrac{a_k}{b_k} = 1$ is not 0 and not ∞, we conclude that $\sum a_k = \sum \dfrac{k}{k^2+1}$ "behaves the same" as $\sum b_k = \sum \dfrac{1}{k}$ which diverges (harmonic series) so $\sum a_k$ DIVERGES.

ANOTHER METHOD: Integral test. $\displaystyle\int_1^{\infty} \frac{x}{x^2+1}\,dx = ?$

Consider $\displaystyle\int_1^{M} \frac{x}{x^2+1}\,dx$. Sub $u = x^2+1 \Rightarrow du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$\text{Integral} = \int_2^{M^2+1} \frac{1}{u}\cdot\frac{1}{2}\,du = \frac{1}{2}\left(\ln|M^2+1| - \ln 2\right) \to \infty$$

when $M \to \infty$

36. Classify the series as absolutely convergent, conditionally convergent, or divergent.

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{3k}$$

Series is alternating because of $(-1)^{k+1}$.

"Positive part" of k^{th} term is $\frac{1}{3k}$.

Could also say k^{th} term of given series is $u_k = \frac{(-1)^{k+1}}{3k}$ and then "positive part" is $|u_k| = \frac{1}{3k}$.

Since $\frac{1}{3k}$ decreases to 0, the given series <u>converges</u> "as is" by the alternating series test.

However, the related series $\sum |u_k| = \sum \frac{1}{3k} = \frac{1}{3} \sum \frac{1}{k}$ <u>diverges</u> (constant multiple of harmonic series).

Therefore the given series is CONDITIONALLY CONVERGENT.

37. Classify the series as absolutely convergent, conditionally convergent, or divergent.

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{4/3}}$$

$\sum u_k$ where $u_k = \dfrac{(-1)^{k+1}}{k^{4/3}}$

Given series is alternating and $\dfrac{1}{k^{4/3}}$ decreases to 0.

Therefore given series converges "as is" by the alternating series test

Related series $\sum |u_k| = \sum \dfrac{1}{k^{4/3}}$

is a p-series with $p = \frac{4}{3} > 1$, so converges.

Since $\sum u_k$ and $\sum |u_k|$ both converge, the given series $\sum u_k = \sum \dfrac{(-1)^{k+1}}{k^{4/3}}$

is ABSOLUTELY CONVERGENT

38. Classify the series as absolutely convergent, conditionally convergent, or divergent.

$$\sum_{k=1}^{\infty} \frac{(-4)^k}{k^2}$$

Careful. Given series is $\sum u_k$

where $u_k = \dfrac{(-4)^k}{k^2} = \dfrac{(-1)^k 4^k}{k^2}$.

So, "positive part" of k^{th} term

is $|u_k| = \dfrac{4^k}{k^2}$ which does NOT decrease to 0.
(To "guess" this,
notice top is exponential
and bottom is polynomial)

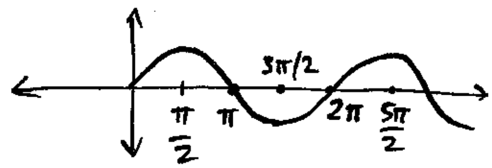
Note $|u_{k+1}| \div |u_k| = \dfrac{4^{k+1}}{(k+1)^2} \cdot \dfrac{k^2}{4^k} = \underbrace{\dfrac{4^{k+1}}{4^k}}_{=4} \cdot \underbrace{\dfrac{k^2}{k^2+2k+1}}_{\to 1}$

So $\dfrac{|u_{k+1}|}{|u_k|} \to 4$ so $|u_{k+1}|$ is LARGER than $|u_k|$

The given series DIVERGES

39. Classify the series as absolutely convergent, conditionally convergent, or divergent.

$$\sum_{k=1}^{\infty} \sin \frac{k\pi}{2}$$



Note: $\sin \frac{\pi}{2} = 1$

$\sin \frac{2\pi}{2} = 0$

$\sin \frac{3\pi}{2} = -1$

$\sin \frac{4\pi}{2} = 0$

etc.

So given series is $1+0+(-1)+0+1+0+(-1)+0+\ldots$

which is the same as

$$1+(-1)+1+(-1)+\ldots$$

which DIVERGES