MAC2312 Section U03
Suggested problems for final exam.
The final exam is **cumulative**.
You should **also** practice the suggested problems for Tests 1 through 3.

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April 12, 2017

1. Determine whether the series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{1}{5k+2}$$

GUESS: When k is large, $\frac{1}{5k+2} \approx \frac{1}{5k}$ so $\sum \frac{1}{5k+2} \approx \sum \frac{1}{5k} = \frac{1}{5}\sum \frac{1}{k}$

$a_k = \frac{1}{5k+2}$. Try LIMIT COMPARISON TEST with $b_k = \frac{1}{k}$.

Then $L = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty}\left(\frac{1}{5k+2} \div \frac{1}{k}\right)$

$= \lim_{k\to\infty}\left(\frac{1}{5k+2}\cdot\frac{k}{1}\right) = \lim_{k\to\infty} \frac{k}{5k+2} \cdot \frac{\frac{1}{k}}{\frac{1}{k}}$

$= \lim_{k\to\infty} \frac{1}{5+\frac{2}{k}} = \frac{1}{5+0} = \frac{1}{5}$ which is not 0, not ∞.

(ROUGH IDEA: $\frac{a_k}{b_k} \to \frac{1}{5}$ so $a_k \approx \frac{1}{5} b_k$)

Therefore $\sum a_k$ and $\underbrace{\sum b_k}_{\text{harmonic}}$ "behave the same" so $\sum a_k$ DIVERGES.

GUESS: When k is large, $\frac{1}{k(k+1)} \approx \frac{1}{k^2}$ so $\sum \frac{1}{k(k+1)} \approx \sum \frac{1}{k^2}$

2. Determine whether the series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$$

METHOD 1: Basic comparison. Notice $k(k+1) > k \cdot k = k^2$

So $\frac{1}{k(k+1)} < \frac{1}{k^2}$ so $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} < \sum_{k=1}^{\infty} \frac{1}{k^2}$.

We know $\sum \frac{1}{k^2}$ is a convergent p-series $(p = 2 > 1)$

so by basic comparison, $\sum \frac{1}{k(k+1)}$ CONVERGES.

METHOD 2: This is one of those rare series where we can find a formula for the n^{th} partial sum! (Telescoping series)

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots + \frac{1}{(n-1)\cdot n} + \frac{1}{n\cdot(n+1)}$$

$$= \frac{2-1}{1\cdot 2} + \frac{3-2}{2\cdot 3} + \frac{4-3}{3\cdot 4} + \dots + \frac{n-(n-1)}{(n-1)\cdot n} + \frac{(n+1)-n}{n\cdot(n+1)}$$

$$= \left(\frac{2}{1\cdot 2} - \frac{1}{1\cdot 2}\right) + \left(\frac{3}{2\cdot 3} - \frac{2}{2\cdot 3}\right) + \dots + \left(\frac{n}{(n-1)n} - \frac{n-1}{(n-1)n}\right) + \left(\frac{n+1}{n(n+1)} - \frac{n}{n(n+1)}\right)$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$= 1 - \frac{1}{n+1}$ which approaches 1 as $n \to \infty$, so series CONVERGES.

GUESS: If k is large, $\frac{k}{1+k^2} \approx \frac{k}{k^2} = \frac{1}{k}$, so $\sum \frac{k}{1+k^2} \approx \sum \frac{1}{k}$

3. Determine whether the series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{k}{1+k^2}$$

METHOD 1: Try limit comparison test. $a_k = \frac{k}{1+k^2}$.

Try $b_k = \frac{k}{k^2} = \frac{1}{k}$. Then $L = \lim\limits_{k\to\infty} \frac{a_k}{b_k}$

$= \lim\limits_{k\to\infty} \left(\frac{k}{1+k^2} \div \frac{1}{k} \right) = \lim\limits_{k\to\infty} \left(\frac{k}{1+k^2} \cdot \frac{k}{1} \right) = \lim\limits_{k\to\infty} \frac{k^2}{1+k^2}$

$= \lim\limits_{k\to\infty} \frac{1}{\frac{1}{k^2}+1} = \frac{1}{0+1} = 1$, which is not 0 and not ∞.

Therefore $\sum a_k$ and $\sum b_k$ "behave the same" so $\sum a_k$ DIVERGES.

METHOD 2: Integral test will work. $\int_1^{\infty} \frac{x}{1+x^2}\,dx$

Consider $\int_1^{M} \frac{x}{1+x^2}\,dx$. Substitute $u = 1+x^2$

$du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$

If $x=1$, then $u = 1+1^2 = 2$

If $x = M$, then $u = 1+M^2$

Integral $= \int_{u=2}^{u=1+M^2} \frac{1}{2} \cdot \frac{1}{u}\,du$

$= \frac{1}{2}\Big[\ln|u| \Big]_2^{1+M^2} = \frac{1}{2}\Big(\underbrace{\ln(1+M^2)}_{\text{DIVERGES}} - \underbrace{\ln 2}_{\text{just a number}} \Big)$

GUESSES? Compare to other known series?

4. Determine whether the series converges or diverges.

$$\sum_{k=2}^{\infty} \frac{\ln k}{k}$$

METHOD 1: Basic comparison. Notice $\ln k > 1$ (if $k > e$).

Then $\frac{\ln k}{k} > \frac{1}{k}$, so $\sum \frac{\ln k}{k} > \sum \frac{1}{k}$.

We know $\sum \frac{1}{k}$ diverges (harmonic series) so by basic comparison,

we conclude $\sum \frac{\ln k}{k}$ <u>diverges.</u>

METHOD 2: Integral test. $\int_2^{\infty} \frac{\ln x}{x}\, dx$

Consider $\int_2^{M} \frac{\ln x}{x}\, dx = \int_{x=2}^{x=M} \ln x \cdot \frac{1}{x}\, dx$

Sub $u = \ln x$, $du = \frac{1}{x}\, dx$ $\quad x = 2 \Rightarrow u = \ln 2$, $x = M \Rightarrow u = \ln M$ $\quad \int_{u=\ln 2}^{u=\ln M} u\, du$

$$= \left[\frac{u^2}{2} \right]_{\ln 2}^{\ln M} = \frac{1}{2}\left((\ln M)^2 - \underbrace{(\ln 2)^2}_{\text{just a number}} \right)$$

which DIVERGES as $m \to \infty$.

GUESS: If k is large then $\frac{1}{\sqrt{k^2+1}} \approx \frac{1}{\sqrt{k^2}} = \frac{1}{k}$

5. Determine whether the series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2+1}}$$

In the given "mystery series", $a_k = \frac{1}{\sqrt{k^2+1}}$.

Try Limit Comparison Test with $b_k = \frac{1}{k}$.

Then $L = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \left(\frac{1}{\sqrt{k^2+1}} \div \frac{1}{k} \right)$

$= \lim_{k\to\infty} \left(\frac{1}{\sqrt{k^2+1}} \cdot \frac{k}{1} \right) = \lim_{k\to\infty} \frac{k}{\sqrt{k^2+1}}$

$= \lim_{k\to\infty} \frac{\sqrt{k^2}}{\sqrt{k^2+1}} = \lim_{k\to\infty} \sqrt{\frac{k^2}{k^2+1}} = \lim_{k\to\infty} \sqrt{\frac{1}{1+\frac{1}{k^2}}}$

$= \sqrt{\frac{1}{1+0}} = \sqrt{1} = 1$ which is not 0, not ∞.

Therefore $\sum a_k$ and $\sum b_k$ "behave the same"

and $\sum b_k = \sum \frac{1}{k}$ is known to diverge, so $\sum a_k$ DIVERGES.

6. Determine whether the series converges or diverges.

$$\sum_{k=2}^{\infty} \frac{1}{\ln k}$$

How to GUESS answer? How can we compare $\frac{1}{\ln k}$ to something else? We know $\ln k$ grows slowly.

We know $\ln k < k$, so $\frac{1}{\ln k} > \frac{1}{k}$.

But then $\sum_{k=2}^{\infty} \frac{1}{\ln k} > \underbrace{\sum_{k=2}^{\infty} \frac{1}{k}}_{\text{The harmonic series (minus the first term)}}$ which diverges

So by the Basic Comparison test,

$\sum \frac{1}{\ln k}$ also DIVERGES.

7. Determine whether the series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{1}{3^k + 5}$$

GUESS: When k is large, $3^k + 5 \approx 3^k$

so $\dfrac{1}{3^k+5} \approx \dfrac{1}{3^k} = \left(\dfrac{1}{3}\right)^k$.

METHOD 1: Basic comparison. $a_k = \dfrac{1}{3^k+5}$

Since $3^k + 5 > 3^k$, we conclude $\dfrac{1}{3^k+5} < \dfrac{1}{3^k} = \left(\dfrac{1}{3}\right)^k$

so $\sum \dfrac{1}{3^k+5} < \sum \left(\dfrac{1}{3}\right)^k$ which is a convergent geometric series $\left(r = \frac{1}{3}\right)$

So by basic comparison, $\sum \dfrac{1}{3^k+5}$ converges.

METHOD 2: Limit comparison test. $a_k = \dfrac{1}{3^k+5}$

Choose $b_k = \dfrac{1}{3^k}$. Then $L = \lim\limits_{k\to\infty} \dfrac{a_k}{b_k} = \lim\limits_{k\to\infty} \left(\dfrac{1}{3^k+5} \div \dfrac{1}{3^k} \right)$

$= \lim\limits_{k\to\infty} \left(\dfrac{1}{3^k+5} \cdot \dfrac{3^k}{1} \right) = \lim\limits_{k\to\infty} \dfrac{3^k}{3^k+5} = \lim\limits_{k\to\infty} \dfrac{1}{1+\frac{5}{3^k}}$

$= \dfrac{1}{1+0} = 1$ not 0, not ∞. Therefore $\sum a_k$ "behaves the same"

as $\sum b_k = \sum \dfrac{1}{3^k} = \sum \left(\dfrac{1}{3}\right)^k$, which converges. (geometric with $r = \frac{1}{3}$)

constant multiple

GUESS: If k is large, then $\frac{1}{(2k+3)^{17}} \approx \frac{1}{(2k)^{17}} = \frac{1}{2^{17}} \cdot \frac{1}{k^{17}}$

8. Determine whether the series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{1}{(2k+3)^{17}}$$

METHOD 1: Basic comparison test. Notice $2k+3 > 2k$

so $(2k+3)^{17} > (2k)^{17}$ so $\frac{1}{(2k+3)^{17}} < \frac{1}{(2k)^{17}} = \frac{1}{2^{17}} \cdot \frac{1}{k^{17}}$

so $\sum \frac{1}{(2k+3)^{17}} < \sum \frac{1}{2^{17}} \cdot \frac{1}{k^{17}} = \frac{1}{2^{17}} \sum \frac{1}{k^{17}}$

which is a constant multiple of a p-series with $p = 17 > 1$ so the "mystery series" is LESS than a convergent series, so $\sum \frac{1}{(2k+3)^{17}}$ CONVERGES by basic comparison.

METHOD 2: Limit comparison test. $a_k = \frac{1}{(2k+3)^{17}}$

Choose $b_k = \frac{1}{k^{17}}$ (could also choose $b_k = \frac{1}{(2k)^{17}}$)

Then $L = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \left(\frac{1}{(2k+3)^{17}} \div \frac{1}{k^{17}} \right) = \lim_{k\to\infty} \left(\frac{1}{(2k+3)^{17}} \cdot \frac{k^{17}}{1} \right)$

$= \lim_{k\to\infty} \frac{k^{17}}{(2k+3)^{17}} \cdot \frac{\frac{1}{k^{17}}}{\frac{1}{k^{17}}} = \lim_{k\to\infty} \frac{1}{\left(\frac{2k+3}{k}\right)^{17}}$

$= \lim_{k\to\infty} \frac{1}{\left(2+\frac{3}{k}\right)^{17}} = \frac{1}{(2+0)^{17}} = \frac{1}{2^{17}}$ not 0, not ∞

So $\sum a_k$ and $\sum b_k$ "behave the same"

$\sum b_k$ is convergent p-series ($p = 17 > 1$)

so $\sum a_k$ converges

9. Determine whether the series converges or diverges.

$$\sum_{k=0}^{\infty} \frac{1}{k!}$$

GUESS: $k!$ grows very quickly, so $\frac{1}{k!}$ approaches 0 quickly so maybe $\sum \frac{1}{k!}$ converges.

For series containing factorials, ratio test often works.

k^{th} term of given series is $u_k = \frac{1}{k!}$ Then the $(k+1)^{th}$ term is $u_{k+1} = \frac{1}{(k+1)!}$. Then $L = \lim_{k\to\infty} \frac{u_{k+1}}{u_k}$

$$= \lim_{k\to\infty}\left(\frac{1}{(k+1)!} \div \frac{1}{k!}\right) = \lim_{k\to\infty}\left(\frac{1}{(k+1)!}\cdot\frac{k!}{1}\right) = \lim_{k\to\infty}\frac{k!}{(k+1)!}$$

$$= \lim_{k\to\infty} \frac{1\cdot 2\cdot 3\cdots(k-1)\cdot k}{1\cdot 2\cdot 3\cdots (k-1)\cdot k\cdot(k+1)} = \lim_{k\to\infty}\frac{1}{k+1} = 0.$$

Since $L = 0 < 1$, ratio test says the series CONVERGES.

Other methods? Since $k!$ is very large, maybe basic comparison will work with a "sloppy" bound.

HINT: Notice

$$\begin{aligned} 2! &= 1\times 2 & &= 2^1 \\ 3! &= 1\times 2\times 3 &< 1\times 2\times 2 &= 2^2 \\ 4! &= 1\times 2\times 3\times 4 &< 1\times 2\times 2\times 2 &= 2^3 \end{aligned}$$

In this way, we can see $k! < 2^{k-1}$ . . .

So $\frac{1}{k!} < \frac{1}{2^{k-1}}$ . . . .

10. Determine whether the series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{k^2}{5^k}$$

GUESS? Denominator is 5^k, exponential, grows very fast.
Numerator k^2 grows much less fast.
Ratio test often works for series containing exponentials.
k^{th} term is $u_k = \dfrac{k^2}{5^k}$. $(k+1)^{th}$ term is $u_{k+1} = \dfrac{(k+1)^2}{5^{k+1}}$.

$$L = \lim_{k\to\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to\infty}\left(\frac{(k+1)^2}{5^{k+1}} \div \frac{k^2}{5^k}\right)$$

$$= \lim_{k\to\infty}\left(\frac{(k+1)^2}{5^{k+1}} \cdot \frac{5^k}{k^2}\right) = \lim_{k\to\infty}\left(\frac{(k+1)^2}{k^2} \cdot \frac{5^k}{5^{k+1}}\right)$$

$$= \lim_{k\to\infty}\left(\frac{k^2+2k+1}{k^2} \cdot \frac{1}{5}\right) = \frac{1}{5} \cdot \lim_{k\to\infty} \frac{k^2+2k+1}{k^2} \cdot \frac{\frac{1}{k^2}}{\frac{1}{k^2}}$$

$$= \frac{1}{5} \cdot \lim_{k\to\infty} \frac{1+\frac{2}{k}+\frac{1}{k^2}}{1} = \frac{1}{5} \cdot \frac{1+0+0}{1} = \frac{1}{5}$$

Since $\lim_{k\to\infty} \dfrac{u_{k+1}}{u_k} = \dfrac{1}{5}$ which is < 1, we conclude that

$\sum \dfrac{k^2}{5^k}$ CONVERGES.

(Other methods? "Sloppy" bounds? $k^2 < 2^k$ so $\dfrac{k^2}{5^k} < \dfrac{2^k}{5^k} < \left(\dfrac{2}{5}\right)^k$)

GUESS: If k is large then $\frac{1}{1+\sqrt{k}} \approx \frac{1}{\sqrt{k}} = \frac{1}{k^{1/2}}$

11. Determine whether the series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{1}{1+\sqrt{k}}$$

Limit comparison test. k^{th} term of given series is $a_k = \frac{1}{1+\sqrt{k}}$. Guess $b_k = \frac{1}{\sqrt{k}} = \frac{1}{k^{1/2}}$.

Then $L = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty}\left(\frac{1}{1+\sqrt{k}} \div \frac{1}{\sqrt{k}}\right)$

$= \lim_{k\to\infty}\left(\frac{1}{1+\sqrt{k}} \cdot \frac{\sqrt{k}}{1}\right) = \lim_{k\to\infty} \frac{\sqrt{k}}{1+\sqrt{k}} \cdot \frac{\frac{1}{\sqrt{k}}}{\frac{1}{\sqrt{k}}}$

$= \lim_{k\to\infty} \frac{1}{\frac{1}{\sqrt{k}}+1} = \frac{1}{0+1} = 1$ which is not 0, not ∞

Since $\lim_{k\to\infty} \frac{a_k}{b_k}$ is not 0 and not ∞, we conclude that $\sum a_k$ and $\sum b_k$ "behave the same". Now note $\sum b_k$

$= \sum \frac{1}{\sqrt{k}} = \sum \frac{1}{k^{1/2}}$ is a p-series with $p = \frac{1}{2} \leq 1$

So $\sum b_k$ diverges. Therefore $\sum a_k = \sum \frac{1}{1+\sqrt{k}}$ also DIVERGES.

GUESS: $\ln k$ approaches infinity slowly and 2^k approaches infinity quickly

12. Determine whether the series converges or diverges.

$$\sum_{k=2}^{\infty} \frac{\ln k}{2^k}$$

Try ratio test. k^{th} term is $u_k = \dfrac{\ln k}{2^k}$. Then $(k+1)^{th}$ term is $u_{k+1} = \dfrac{\ln(k+1)}{2^{k+1}}$. Then $L = \lim\limits_{k\to\infty} \dfrac{u_{k+1}}{u_k}$

$$= \lim_{k\to\infty}\left(\frac{\ln(k+1)}{2^{k+1}} \div \frac{\ln k}{2^k}\right) = \lim_{k\to\infty}\left(\frac{\ln(k+1)}{2^{k+1}} \cdot \frac{2^k}{\ln k}\right)$$

$$= \lim_{k\to\infty}\left(\frac{2^k}{2^{k+1}} \cdot \frac{\ln(k+1)}{\ln k}\right) = \lim_{k\to\infty}\left(\frac{1}{2} \cdot \frac{\ln(k+1)}{\ln k}\right)$$

$$= \frac{1}{2} \cdot \lim_{k\to\infty} \frac{\ln(k+1)}{\ln k}$$ This limit has the form $\frac{\infty}{\infty}$ so we can use L'HOPITAL'S RULE

$$= \frac{1}{2} \lim_{k\to\infty} \frac{\frac{1}{k+1}}{\frac{1}{k}} = \frac{1}{2} \lim_{k\to\infty}\left(\frac{1}{k+1} \div \frac{1}{k}\right) = \frac{1}{2} \lim_{k\to\infty}\left(\frac{1}{k+1} \cdot \frac{k}{1}\right)$$

$$= \frac{1}{2} \lim_{k\to\infty} \frac{k}{k+1} = \frac{1}{2} \cdot 1 < 1.$$ Since $\lim\limits_{k\to\infty} \dfrac{u_{k+1}}{u_k} < 1$, the series CONVERGES.

13. Determine whether the series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{3k}$$

k^{th} term is $u_k = \dfrac{(-1)^{k+1}}{3k}$

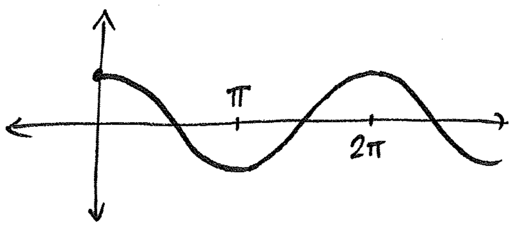
This is an ALTERNATING series. The "positive part" of the k^{th} term is $|u_k| = \dfrac{1}{3k}$ which decreases and approaches 0. Therefore, by the alternating series test, this alternating series CONVERGES.

14. Determine whether the series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{\cos(k\pi)}{k}$$

This is actually an alternating series in disguise!

What is $\cos(k\pi)$? $\cos\pi, \cos 2\pi, \cos 3\pi, \ldots$



$\cos\pi = -1$
$\cos 2\pi = 1$
$\cos 3\pi = -1$
$\cos 4\pi = 1$

$$\cos(k\pi) = (-1)^k$$

$$\sum \frac{\cos(k\pi)}{k} = \sum \frac{(-1)^k}{k}$$ Alternating series

"Positive part" of k^{th} term is $\frac{1}{k}$

which decreases and approaches 0.

Therefore, by the Alternating Series Test, this alternating series CONVERGES.

15. Determine whether the series converges absolutely, converges conditionally, or diverges.

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

k^{th} term of this series is $u_k = \dfrac{(-1)^k}{k}$

Does the series $\sum u_k$ converge "as is"?

Notice $\sum u_k = \sum \dfrac{(-1)^k}{k}$ is an alternating series and the "positive part" of the k^{th} term $|u_k| = \dfrac{1}{k}$ decreases to 0. Therefore by the alternating series test, the series $\sum u_k$ converges "as is".

Now, does the related series $\sum |u_k|$ converge or diverge?

$\sum |u_k| = \sum \left| \dfrac{(-1)^k}{k} \right| = \sum \dfrac{1}{k}$ which is the HARMONIC series which diverges.

That is, $\sum u_k$ converges "as is" but $\sum |u_k|$ diverges.

Therefore $\sum u_k = \sum \dfrac{(-1)^k}{k}$ CONVERGES CONDITIONALLY.

16. Determine whether the series converges absolutely, converges conditionally, or diverges.

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{4/3}}$$

k^{th} term of this series is $u_k = \dfrac{(-1)^k}{k^{4/3}}$

Does the series $\sum u_k = \sum \dfrac{(-1)^k}{k^{4/3}}$ converge "as is"?

Notice $\sum u_k = \sum \dfrac{(-1)^k}{k^{4/3}}$ is an alternating series and the "positive part" of the k^{th} term is $|u_k| = \dfrac{1}{k^{4/3}}$ which decreases to 0. Therefore by the alternating series test, the series $\sum u_k$ converges "as is".

Now, does the related series $\sum |u_k|$ converge or diverge?

$\sum |u_k| = \sum \left| \dfrac{(-1)^k}{k^{4/3}} \right| = \sum \dfrac{1}{k^{4/3}}$ is a p-series with $p = \frac{4}{3} > 1$. This series converges.

That is, $\sum u_k$ converges "as is" and $\sum |u_k|$ also converges.

Therefore $\sum u_k = \sum \dfrac{(-1)^k}{k^{4/3}}$ CONVERGES ABSOLUTELY.

17. Determine whether the series converges absolutely, converges conditionally, or diverges.

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$$

k^{th} term of this series is $u_k = \dfrac{(-1)^k}{k!}$

Does the series $\sum u_k = \sum \dfrac{(-1)^k}{k!}$ converge "as is"?

Notice $\sum u_k = \sum \dfrac{(-1)^k}{k!}$ is an alternating series and the "positive part" of the k^{th} term is $|u_k| = \dfrac{1}{k!}$ which decreases to 0. Therefore by the alternating series test, the series $\sum u_k$ converges "as is".

Now, does the related series $\sum |u_k|$ converge or diverge?

$\sum |u_k| = \sum \left| \dfrac{(-1)^k}{k!} \right| = \sum \dfrac{1}{k!}$ This series appeared earlier Question 9.

The series $\sum \dfrac{1}{k!}$ can be shown to converge by the ratio test.

That is, $\sum u_k$ converges "as is" and $\sum |u_k|$ also converges.

Therefore $\sum u_k = \sum \dfrac{(-1)^k}{k!}$ CONVERGES ABSOLUTELY.

18. Determine whether the series converges absolutely, converges conditionally, or diverges.

$$\sum_{k=2}^{\infty} \frac{(-1)^k \ln k}{k}$$

k^{th} term of this series is $u_k = \dfrac{(-1)^k \ln k}{k}$

Does the series $\sum u_k = \sum \dfrac{(-1)^k \ln k}{k}$ converge "as is"?

Notice $\sum u_k = \sum \dfrac{(-1)^k \ln k}{k}$ is an alternating series and the "positive part" of the k^{th} term is $|u_k| = \dfrac{\ln k}{k}$ which decreases to 0. (Why? Roughly speaking, because $\ln k$ approaches ∞ more slowly than k does. Can also use L'Hopital's rule: $\lim\limits_{k\to\infty} \dfrac{\ln k}{k} = \lim\limits_{k\to\infty} \dfrac{\frac{1}{k}}{1} = 0.$)

So by the alternating series test, the series $\sum u_k$ converges "as is".

Next, does the related series $\sum |u_k| = \sum \frac{\ln k}{k}$ converge or diverge? That series appeared earlier (Question 4). The series $\sum \frac{\ln k}{k}$ can be shown to diverge either by basic comparison or by the integral test.

So, $\sum u_k$ converges "as is" but $\sum |u_k|$ diverges.

Therefore $\sum u_k = \sum \dfrac{(-1)^k \ln k}{k}$ CONVERGES CONDITIONALLY.

19. Determine whether the series converges absolutely, converges conditionally, or diverges.

$$\sum_{k=0}^{\infty} \left(\frac{-3}{5}\right)^k$$

k^{th} term of this series is $u_k = \left(-\frac{3}{5}\right)^k = (-1)^k \left(\frac{3}{5}\right)^k$.

Does the series $\sum u_k = \sum \left(-\frac{3}{5}\right)^k$ converge "as is"?

We could either use the alternating series test, or observe that $\sum u_k = \sum \left(-\frac{3}{5}\right)^k$ is a geometric series with $r = \frac{-3}{5}$.

Since $-1 < r < 1$, the series $\sum \left(-\frac{3}{5}\right)^k$ converges.

Next, does the related series $\sum |u_k| = \sum \left(\frac{3}{5}\right)^k$ converge?

The series $\sum \left(\frac{3}{5}\right)^k$ is a geometric series with $r = \frac{3}{5}$.

Since $-1 < r < 1$, the series $\sum \left(\frac{3}{5}\right)^k$ converges.

That is, $\sum u_k = \sum \left(-\frac{3}{5}\right)^k$ converges "as is", and the series $\sum |u_k| = \sum \left(\frac{3}{5}\right)^k$ also converges.

So $\sum u_k = \sum \left(-\frac{3}{5}\right)^k$ CONVERGES ABSOLUTELY.

20. Find the Maclaurin series for the function.

$$f(x) = e^{-x}$$

METHOD 1: Find k^{th} derivative.

$$\left.\begin{aligned} f(x) &= e^{-x} \\ f'(x) &= -e^{-x} \\ f''(x) &= e^{-x} \\ f'''(x) &= -e^{-x} \end{aligned}\right\} \Rightarrow \begin{aligned} f(0) &= 1 \\ f'(0) &= -1 \\ f''(0) &= 1 \\ f'''(0) &= -1 \end{aligned}$$

etc. etc.

General Maclaurin Series is $f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$

which in this case becomes $1 + \frac{-1}{1!}x + \frac{1}{2!}x^2 + \frac{-1}{3!}x^3 + \ldots$

which can also be written $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k$

METHOD 2: Start with known MacLaurin series and alter it.

We know $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$

If we replace x with $-x$, we get:

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!}$$

21. Find the Maclaurin series for the function.

$$f(x) = e^{5x}$$

METHOD 1: Find k^{th} derivative.

$$\left.\begin{aligned} f(x) &= e^{5x} \\ f'(x) &= 5e^{5x} \\ f''(x) &= 5^2 e^{5x} \\ f'''(x) &= 5^3 e^{5x} \\ &\text{etc.} \end{aligned}\right\} \Rightarrow \begin{aligned} f(0) &= 1 \\ f'(0) &= 5 \\ f''(0) &= 5^2 \\ f'''(0) &= 5^3 \\ &\text{etc.} \end{aligned}$$

General MacLaurin series is $f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$

which in this case becomes $1 + \frac{5}{1!}x + \frac{5^2}{2!}x^2 + \frac{5^3}{3!}x^3 + \ldots$

Which can also be written $\sum_{k=0}^{\infty} \frac{5^k x^k}{k!}$.

METHOD 2: Start with known MacLaurin series and alter it.

We know $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

If we replace x with 5x, we get:

$$e^{5x} = 1 + 5x + \frac{5^2x^2}{2!} + \frac{5^3x^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{5^k x^k}{k!}$$

22. Find the Maclaurin series for the function.

$$f(x) = \frac{1}{1+x}$$

METHOD 1: Find k^{th} derivative.

$$\left.\begin{aligned} f(x) &= (1+x)^{-1} \\ f'(x) &= (-1)(1+x)^{-2} \\ f''(x) &= (-1)(-2)(1+x)^{-3} \\ f'''(x) &= (-1)(-2)(-3)(1+x)^{-4} \end{aligned}\right\} \Rightarrow \begin{aligned} f(0) &= 1 \\ f'(0) &= -1 \\ f''(0) &= +2! \\ f'''(0) &= -3! \end{aligned}$$

etc. etc.

$$f^{(k)}(0) = (-1)^k k!$$

General MacLaurin series is $f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$

which in this case becomes $1 + \frac{-1!}{1!}x + \frac{2!}{2!}x^2 + \frac{-3!}{3!}x^3 + \ldots$

$$= 1 - x + x^2 - x^3 + \ldots = \sum_{k=0}^{\infty} (-1)^k x^k$$

METHOD 2: We know the formula for the sum of a geometric series.

$a + ar + ar^2 + ar^3 + \ldots = \frac{a}{1-r}$. Then choose $a = 1$, $r = -x$

$$1 - x + x^2 - x^3 + \ldots = \frac{1}{1-(-x)} = \frac{1}{1+x}$$

(Note: Converges for only <u>some</u> values of x)

23. Find the Maclaurin series for the function.

$$f(x) = \ln(1 + x)$$

METHOD 1: Find k^{th} derivative.

$$\left.\begin{aligned} f(x) &= \ln(1+x) \\ f'(x) &= \frac{1}{1+x} = (1+x)^{-1} \\ f''(x) &= (-1)(1+x)^{-2} \\ f'''(x) &= (-1)(-2)(1+x)^{-3} \\ f^{(4)}(x) &= (-1)(-2)(-3)(1+x)^{-4} \end{aligned}\right\} \Rightarrow \begin{aligned} f(0) &= \ln(1) = 0 \\ f'(0) &= 1 \\ f''(0) &= -1 \\ f'''(0) &= +2! \\ f^{(4)}(0) &= -3! \end{aligned}$$

etc.

So MacLaurin series is $f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$

$$= 0 + \frac{1}{1!}x + \frac{-1!}{2!}x^2 + \frac{2!}{3!}x^3 + \frac{-3!}{4!}x^4 + \ldots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

which can be written in sigma notation as $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k}$

or $\sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1}$ or other correct ways

METHOD 2: Start with series for $\frac{1}{1+x}$ and take antiderivative!

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$$

take antideriv

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + C$$

By plugging in $x=0$ we find that $C=0$

24. Find the Maclaurin series for the function.

$$f(x) = (1+x)^{1/4}.$$

Try to find k^{th} derivative. First few derivatives are:

$$\left.\begin{aligned} f(x) &= (1+x)^{1/4} \\ f'(x) &= \frac{1}{4}(1+x)^{-3/4} \\ f''(x) &= \frac{1}{4}\cdot\frac{-3}{4}(1+x)^{-7/4} \\ f'''(x) &= \frac{1}{4}\cdot\frac{-3}{4}\cdot\frac{-7}{4}(1+x)^{-11/4} \end{aligned}\right\} \Rightarrow \begin{aligned} f(0) &= 1 \\ f'(0) &= \frac{1}{4} \\ f''(0) &= \frac{-1\cdot 3}{4^2} = \frac{-3}{16} \\ f'''(0) &= \frac{+1\cdot 3\cdot 7}{4^3} = \frac{21}{64} \end{aligned}$$

Not obvious how to write a general expression for $f^{(k)}(0)$, but we have $f^{(4)}(0) = \frac{-1\cdot 3\cdot 7\cdot 11}{4^4}$, $f^{(5)}(0) = \frac{+1\cdot 3\cdot 7\cdot 11\cdot 15}{4^5}$, etc.

MacLaurin series is $f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$

$$= 1 + \frac{1/4}{1}x + \frac{-3/16}{2}x^2 + \frac{21/64}{6}x^3 + \dots$$

$$= 1 + \frac{1}{4}x - \frac{3}{32}x^2 + \frac{7}{128}x^3 + \dots$$

25. Find the Maclaurin series for the function.

$$f(x) = (1 + 2x)^{1/3}.$$

Try to find k^{th} derivative. First few derivatives are:

$$\left.\begin{aligned} f(x) &= (1+2x)^{1/3} \\ f'(x) &= \frac{1}{3}(1+2x)^{-2/3}\cdot 2 \\ f''(x) &= \frac{1}{3}\cdot\frac{-2}{3}(1+2x)^{-5/3}\cdot 2^2 \\ f'''(x) &= \frac{1}{3}\cdot\frac{-2}{3}\cdot\frac{-5}{3}(1+2x)^{-8/3}\cdot 2^3 \end{aligned}\right\} \Rightarrow \begin{aligned} f(0) &= 1 \\ f'(0) &= \frac{1}{3}\cdot 2 = \frac{2}{3} \\ f''(0) &= \frac{-1\cdot 2}{3^2}\cdot 2^2 = \frac{-8}{9} \\ f'''(0) &= \frac{+1\cdot 2\cdot 5}{3^3}\cdot 2^3 = \frac{80}{27} \end{aligned}$$

Not obvious how to write a <u>general</u> expression for $f^{(k)}(0)$ but we have $f^{(4)}(0) = \frac{-1\cdot 2\cdot 5\cdot 8}{3^4}\cdot 2^4$, $f^{(5)}(0) = \frac{+1\cdot 2\cdot 5\cdot 8\cdot 11}{3^5}\cdot 2^5$, etc.

MacLaurin series is $f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$

$$= 1 + \frac{2/3}{1}x + \frac{-8/9}{2}x^2 + \frac{80/27}{6}x^3 + \ldots$$

$$= 1 + \frac{2}{3}x - \frac{4}{9}x^2 + \frac{40}{81}x^3 + \ldots$$

26. Find the Maclaurin series for the function.

$$f(x) = xe^x$$

METHOD 1: Find k^{th} derivative.

$$\left.\begin{aligned} f(x) &= xe^x \\ f'(x) &= 1e^x + xe^x = (1+x)e^x \\ f''(x) &= 1e^x + (1+x)e^x = (2+x)e^x \\ f'''(x) &= 1e^x + (2+x)e^x = (3+x)e^x \end{aligned}\right\} \Rightarrow \begin{aligned} f(0) &= 0 \\ f'(0) &= 1 \\ f''(0) &= 2 \\ f'''(0) &= 3 \end{aligned}$$

etc. $\qquad$ etc.

$$f^{(k)}(0) = k$$

So MacLaurin series is $f(0) + \dfrac{f'(0)}{1!}x + \dfrac{f''(0)}{2!}x^2 + \dfrac{f'''(0)}{3!}x^3 + \ldots$

$$= 0 + \frac{1}{1!}x + \frac{2}{2!}x^2 + \frac{3}{3!}x^3 + \ldots = \sum_{k=0}^{\infty} \frac{k}{k!}x^k$$

$$= \frac{x}{0!} + \frac{x^2}{1!} + \frac{x^3}{2!} + \ldots = \sum_{k=1}^{\infty} \frac{x^k}{(k-1)!} \text{ or } \sum_{k=0}^{\infty} \frac{x^{k+1}}{k!}$$

METHOD 2: Start with "known" series for e^x and alter it.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$xe^x = x \cdot \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{x \cdot x^k}{k!} = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k!}$$

27. Find the Maclaurin series for the function.

$$f(x) = x \sin x$$

METHOD 1: Find k^{th} derivative.

$f(x) = x \sin x$

$f'(x) = 1 \sin x + x \cos x$

$f''(x) = \cos x + 1 \cos x - x \sin x$
$= 2\cos x - x \sin x$

$f'''(x) = -2\sin x - 1\sin x - x\cos x$
$= -3\sin x - x\cos x$

$f^{(4)}(x) = -3\cos x - 1\cos x + x\sin x$
$= -4\cos x + x\sin x$

$\Rightarrow$

$f(0) = 0$

$f'(0) = 0$

$f''(0) = 2$

$f'''(0) = 0$

$f^{(4)}(0) = -4$

$f^{(5)}(0) = 0$

$f^{(6)}(0) = +6$

$f^{(5)}(x) = 5\sin x + x\cos x$

$f^{(6)}(x) = 6\cos x - x\sin x$

etc.

MacLaurin series is $f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$

$$= 0 + \frac{0}{1!}x + \frac{2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{-4}{4!}x^4 + \frac{0}{5!}x^5 + \frac{6}{6!}x^6 + \cdots$$

$$= \frac{2}{2!}x^2 - \frac{4}{4!}x^4 + \frac{6}{6!}x^6 - \frac{8}{8!}x^8 + \cdots = \frac{x^2}{1!} - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \cdots$$

METHOD 2: Start with "known" series for $\sin x$ and alter it.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \Rightarrow x\sin x = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \cdots$$

28. Find the Maclaurin polynomial of degree 2 for $f(x) = \cos x$, and use it to estimate $\cos(0.1)$ to the nearest thousandth.

$$\left.\begin{aligned} f(x) &= \cos x \\ f'(x) &= -\sin x \\ f''(x) &= -\cos x \end{aligned}\right\} \Rightarrow \begin{aligned} f(0) &= 1 \\ f'(0) &= 0 \\ f''(0) &= -1 \end{aligned}$$

Deg 2 Maclaurin polynomial is $f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2$

$$= 1 + \frac{0}{1!}x + \frac{-1}{2!}x^2 = 1 - \frac{x^2}{2}$$

So $\cos x \approx 1 - \frac{x^2}{2}$ if x is near 0

So $\cos(0.1) \approx 1 - \frac{(0.1)^2}{2} = 1 - \frac{0.01}{2}$

$$= 1 - 0.005 = \underline{0.995}$$

If you're curious, a better approximation using a computer

is $\cos(0.1) \approx 0.995004165$

29. Find the Maclaurin polynomial of degree 3 for $f(x) = \ln(1+x)$, and use it to estimate $\ln(1.1)$ to the nearest ten thousandth.

$$\left.\begin{aligned} f(x) &= \ln(1+x) \\ f'(x) &= \frac{1}{1+x} = (1+x)^{-1} \\ f''(x) &= -1(1+x)^{-2} \\ f'''(x) &= +2(1+x)^{-3} \end{aligned}\right\} \Rightarrow \begin{aligned} f(0) &= \ln 1 = 0 \\ f'(0) &= 1 \\ f''(0) &= -1 \\ f'''(0) &= 2 \end{aligned}$$

Deg 3 MacLaurin polynomial is $f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$

$$= 0 + \frac{1}{1!}x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 = x - \frac{x^2}{2} + \frac{x^3}{3}$$

$$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3}$$

$$\ln(1.1) = \ln(1+0.1) \approx 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3}$$

$$= 0.1 - \frac{0.01}{2} + \frac{0.001}{3} = 0.1 - 0.005 + 0.000333\ldots$$

$$= 0.095333\ldots$$

Nearest ten thousandth: 0.0953

(BTW, computer says 0.0953101798)