WRITE YOUR NAME:

MAC 2312 Homework 3
Due in class, Friday February 9th
You can use more paper if necessary, but please STAPLE

Question 1. Evaluate the integral.

$$\int_{\pi/12}^{\pi/9} \sec^2 3x \, dx$$

Try substituting $u = 3x$ $\quad x = \frac{\pi}{12} \Rightarrow u = \frac{\pi}{4}$

$du = 3dx$

$\frac{1}{3}du = dx$ $\quad x = \frac{\pi}{9} \Rightarrow u = \frac{\pi}{3}$

$$\int_{x=\pi/12}^{x=\pi/9} \sec^2 3x \, dx = \int_{u=\pi/4}^{u=\pi/3} \sec^2 u \cdot \frac{1}{3} du$$

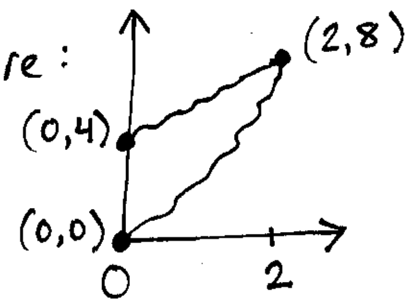
$$= \frac{1}{3}\int_{\pi/4}^{\pi/3} \sec^2 u \, du = \frac{1}{3}\Big[\tan u\Big]_{\pi/4}^{\pi/3}$$

$$= \frac{1}{3}\left(\tan\frac{\pi}{3} - \tan\frac{\pi}{4}\right) = \frac{1}{3}\left(\sqrt{3} - 1\right)$$

or $\dfrac{\sqrt{3}-1}{3}$

Question 2. Notice that the curves $y = x^2 + 4$ and $y = x^3$ intersect when $x = 2$. Let A be the region bounded by those two curves and the y-axis. Find the volume obtained by revolving the region A around the x-axis.

"Lying" picture:

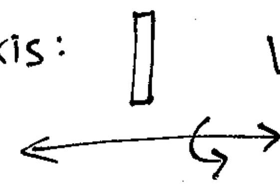


Note: if $0 \le x < 2$ then x^2+4 has bigger output than x^3.

x^2+4 is "top", x^3 is "bottom"

$y = f(x) \Rightarrow$

Revolve around x-axis:



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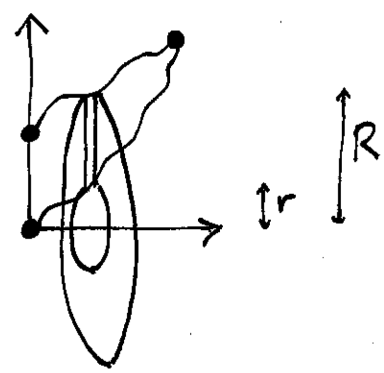
$$\text{Volume} = \int (\pi R^2 - \pi r^2)\,dx.$$
$$= \pi \int (R^2 - r^2)\,dx$$



$$R = \underset{y=x^2+4}{(\text{top curve})} - \underset{y=0}{(\text{axis of rev.})} = x^2+4$$
$$r = \underset{y=x^3}{(\text{bottom curve})} - \underset{y=0}{(\text{axis of rev.})} = x^3$$

$$\text{Volume} = \pi \int_{x=0}^{x=2} \left((x^2+4)^2 - (x^3)^2\right) dx = \pi \int_0^2 (x^4 + 8x^2 + 16 - x^6)\,dx$$

$$= \pi \left[\frac{x^5}{5} + \frac{8x^3}{3} + 16x - \frac{x^7}{7} \right]_0^2 = \pi \left(\frac{2^5}{5} + \frac{8 \cdot 2^3}{3} + 16 \cdot 2 - \frac{2^7}{7} \right)$$

Admittedly, this would take a long time to simplify by hand.
Answer turns out to be $4352\pi/105$

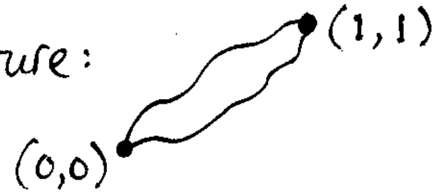
Question 3. Find the volume of the solid that results when the region enclosed by $y = x^2$ and $y = x^3$ is revolved around the line $y = -1$.

Intersection points? Can be found algebraically.

$$x^2 = x^3$$
$$0 = x^3 - x^2$$
$$0 = x^2(x-1)$$
$$x = 0 \text{ or } x = 1$$

Can also "guess" that they intersect at $x=0$ and $x=1$ because we "know" the functions.

"Lying" picture:



Note: if $0 < x < 1$, e.g. $x = \frac{1}{2}$ then $x^2 > x^3$.
x^2 is "top", x^3 is "bottom".

$y = f(x) \Rightarrow$ ▯ $\overset{\leftrightarrow}{dx}$ Revolve around $y = -1$, a horizontal line.

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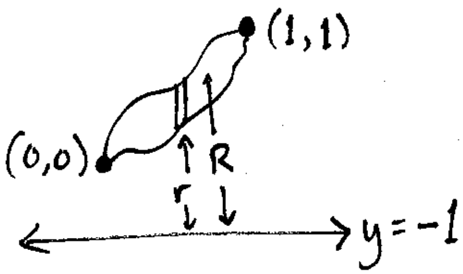

$$R = \underbrace{(\text{top curve})}_{y=x^2} - \underbrace{(\text{axis of rev.})}_{y=-1} = x^2 - (-1) = x^2 + 1$$

$$r = \underbrace{(\text{bottom curve})}_{y=x^3} - \underbrace{(\text{axis of rev.})}_{y=-1} = x^3 - (-1) = x^3 + 1$$

$$\text{Volume} = \pi \int (R^2 - r^2)\,dx = \pi \int_0^1 \left((x^2+1)^2 - (x^3+1)^2\right) dx$$

$$= \pi \int_0^1 \left(x^4 + 2x^2 + 1 - (x^6 + 2x^3 + 1)\right) dx = \pi \int_0^1 (x^4 + 2x^2 - x^6 - 2x^3)\,dx$$

$$= \pi \left[\frac{x^5}{5} + \frac{2x^3}{3} - \frac{x^7}{7} - \frac{2x^4}{4} \right]_0^1 = \pi \left(\frac{1}{5} + \frac{2}{3} - \frac{1}{7} - \frac{1}{2} \right)$$

$$= \pi \left(\frac{42}{210} + \frac{140}{210} - \frac{30}{210} - \frac{105}{210} \right) = \frac{47\pi}{210}$$