WRITE YOUR NAME:

MAC 2312 Homework 5
Due in class, Friday April 6th
You can use more paper if necessary, but please STAPLE

Question 1. Evaluate the integral.

$$\int \sin^3 x \cos^3 x \, dx$$

$$\int \sin^3 x \cos^2 x \underbrace{\cos x \, dx}_{\text{"candidate" for } du}$$

$$= \int \sin^3 x \left(1 - \sin^2 x\right) \cos x \, dx$$

Sub $u = \sin x$
$\Rightarrow du = \cos x \, dx$

$$\text{Integral} = \int u^3 \left(1 - u^2\right) du = \int \left(u^3 - u^5\right) du$$

$$= \frac{u^4}{4} - \frac{u^6}{6} + C = \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + C$$

$$\sin^2 x + \cos^2 x = 1$$

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\tan^2 x + 1 = \sec^2 x$$

Question 2. Evaluate the integral.

$$\int \tan^5 x \sec^4 x \, dx$$

$$\int \tan^5 x \sec^2 x \underbrace{\sec^2 x \, dx}_{\text{"candidate" for } du}$$

$$\int \tan^5 x \cdot (\tan^2 x + 1) \sec^2 x \, dx$$

Sub $u = \tan x$

$\Rightarrow du = \sec^2 x \, dx$

$$\text{Integral} = \int u^5 (u^2 + 1) \, du = \int (u^7 + u^5) \, du$$

$$= \frac{u^8}{8} + \frac{u^6}{6} + C$$

$$= \frac{\tan^8 x}{8} + \frac{\tan^6 x}{6} + C$$

Question 3. Evaluate the integral.

$$\int_3^6 \frac{\sqrt{x^2-9}}{x}\,dx$$

Looks like trig sub. Remember $1-\sin^2\theta = \cos^2\theta$

$1+\tan^2\theta = \sec^2\theta$

$\sec^2\theta - 1 = \tan^2\theta$

We see $\sqrt{x^2-3^2}$ so try $x = 3\sec\theta$

$\Rightarrow dx = 3\sec\theta\tan\theta\,d\theta$

$x=3 \Rightarrow 3\sec\theta = 3 \Rightarrow \sec\theta = 1 \Rightarrow \cos\theta = 1 \Rightarrow \theta = 0$

$x=6 \Rightarrow 3\sec\theta = 6 \Rightarrow \sec\theta = 2 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

$$\text{Integral} = \int_{\theta=0}^{\theta=\pi/3} \frac{\sqrt{9\sec^2\theta-9}}{\cancel{3\sec\theta}}\,\cancel{3\sec\theta}\tan\theta\,d\theta$$

$$= \int_0^{\pi/3} \sqrt{9\tan^2\theta}\,\tan\theta\,d\theta = \int_0^{\pi/3} 3\tan\theta\cdot\tan\theta\,d\theta$$

$$= \int_0^{\pi/3} 3\tan^2\theta\,d\theta = 3\int_0^{\pi/3}(\sec^2\theta - 1)\,d\theta$$

$$= 3\left[\tan\theta - \theta\right]_0^{\pi/3} = 3\Big(\underbrace{\tan\frac{\pi}{3}}_{\sqrt{3}} - \frac{\pi}{3} - \underbrace{\tan 0}_{0} + 0\Big)$$

$$= 3\left(\sqrt{3} - \frac{\pi}{3}\right) \text{ or } 3\sqrt{3} - \pi$$

Question 4. Evaluate the integral.

$$\int_8^{16} \frac{1}{x^2 - 6x - 7}\, dx$$

$$\int_8^{16} \frac{1}{(x+1)(x-7)}\, dx$$

Can use PARTIAL FRACTIONS

$$\frac{1}{(x+1)(x-7)} = \frac{A}{x+1} + \frac{B}{x-7}$$

$$1 = A(x-7) + B(x+1)$$

$$\Rightarrow \dots \Rightarrow A = -\frac{1}{8}, \quad B = \frac{1}{8}$$

Clever trick: $\displaystyle \frac{1}{8}\int_8^{16} \frac{8}{(x+1)(x-7)}\, dx = \frac{1}{8}\int_8^{16} \frac{(x+1)-(x-7)}{(x+1)(x-7)}\, dx$

$$= \frac{1}{8}\int_8^{16} \left(\frac{x+1}{(x+1)(x-7)} - \frac{x-7}{(x+1)(x-7)} \right) dx$$

$$= \frac{1}{8}\int_8^{16} \left(\frac{1}{x-7} - \frac{1}{x+1} \right) dx$$

$$= \frac{1}{8}\Big[\ln|x-7| - \ln|x+1| \Big]_8^{16}$$

$$= \frac{1}{8}\Big((\ln 9 - \ln 17) - (\underbrace{\ln 1}_{=0} - \ln 9) \Big) = \frac{1}{8}\left(2\ln 9 - \ln 17 \right)$$

Question 5. Determine whether the improper integral converges or diverges.

$$\int_2^\infty \frac{1}{x\sqrt{\ln x}}\,dx$$

Consider $\displaystyle\int_2^M \frac{1}{x\sqrt{\ln x}}\,dx = \int_2^M \frac{1}{\sqrt{\ln x}}\cdot\frac{1}{x}\,dx$

Sub $u = \ln x$ $\qquad x = 2 \Rightarrow u = \ln 2$

$\Rightarrow du = \frac{1}{x}\,dx$ $\qquad x = M \Rightarrow u = \ln M$

$$\text{Integral} = \int_{u=\ln 2}^{u=\ln M} \frac{1}{\sqrt{u}}\cdot du = \int_{\ln 2}^{\ln M} u^{-1/2}\,du$$

$$= \left[\frac{u^{1/2}}{1/2}\right]_{\ln 2}^{\ln M} = 2\left[u^{1/2}\right]_{\ln 2}^{\ln M}$$

$$= 2\Big(\underbrace{\sqrt{\ln M}}_{\to\infty} - \underbrace{\sqrt{\ln 2}}_{\text{just a number}}\Big)$$

DIVERGES

Also $\displaystyle\int \frac{1}{\sqrt{u}}\,du$ is "similar" to p-series $\displaystyle\sum\frac{1}{\sqrt{k}} = \sum\frac{1}{k^{1/2}}$

$p = \frac{1}{2} \leq 1$