WRITE YOUR NAME:

MAC 2312 Homework 6
Due in class, Friday March 31st
You can use more paper if necessary, but please STAPLE

Question 1. Evaluate the integral.

$$\int \frac{x^2+4x-9}{x^3-6x^2+9x}\,dx$$

Looks like partial fractions. $x^3-6x^2+9x = x(x^2-6x+9)$
$= x(x-3)^2$.

$$\frac{x^2+4x-9}{x(x-3)^2} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$$

"repeated factor"

multiply both sides by $x(x-3)^2$

$$x^2+4x-9 = A(x-3)^2 + Bx(x-3) + Cx$$
$$= A(x^2-6x+9) + B(x^2-3x) + Cx$$
$$= (A+B)x^2 + (-6A-3B+C)x + 9A$$

$$\Rightarrow \left.\begin{aligned} 1 &= A+B \\ 4 &= -6A-3B+C \\ -9 &= 9A \end{aligned}\right\} \Rightarrow A=-1,\ B=2,\ C=4$$

$$\int \frac{x^2+4x-9}{x(x-3)^2}\,dx = \int\left(\frac{-1}{x} + \frac{2}{x-3} + \frac{4}{(x-3)^2}\right)dx$$

$$= -\int \frac{1}{x}\,dx + 2\int \frac{1}{x-3}\,dx + 4\int (x-3)^{-2}\,dx$$

$$= -\ln|x| + 2\ln|x-3| + 4\frac{(x-3)^{-1}}{-1} + C$$

or $-\ln|x| + 2\ln|x-3| - \frac{4}{x-3} + C$

Question 2. Evaluate the integral.

$$\int \frac{5x^2 + x + 3}{x^3 + x}\,dx$$

Partial fractions. $x^3 + x = x(x^2+1)$

x^2+1 is irreducible of degree 2. Degree of top should be one less than degree of bottom.

$$\frac{5x^2+x+3}{x^3+x} = \frac{5x^2+x+3}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

multiply both sides by $x(x^2+1)$

$$5x^2+x+3 = A(x^2+1) + (Bx+C)x$$
$$= Ax^2 + A + Bx^2 + Cx$$

$$\Rightarrow \left.\begin{array}{l} 5 = A+B \\ 1 = C \\ 3 = A \end{array}\right\} \Rightarrow A=3,\ B=2,\ C=1$$

$$\int \frac{5x^2+x+3}{x^3+x}\,dx = \int \frac{5x^2+x+3}{x(x^2+1)}\,dx = \int \left(\frac{3}{x} + \frac{2x+1}{x^2+1}\right)dx$$

$$= \int \left(\frac{3}{x} + \frac{2x}{x^2+1} + \frac{1}{x^2+1}\right)dx$$

$$= 3\int \frac{1}{x}\,dx + \underbrace{\int \frac{2x}{x^2+1}\,dx}_{\substack{\text{Sub } u = x^2+1 \\ \Rightarrow du = 2x\,dx}} + \underbrace{\int \frac{1}{x^2+1}\,dx}_{\substack{\text{NOTHING TO DO} \\ \text{WITH LOGARITHMS}}}$$

$$= 3\ln|x| + \ln(x^2+1) + \arctan x + C$$

Question 3. Evaluate the integral.

$$\int_4^6 \frac{15x-31}{x^2-4x+3}\,dx$$

Partial fractions. $x^2-4x+3 = (x-1)(x-3)$

$$\frac{15x-31}{x^2-4x+3} = \frac{15x-31}{(x-1)(x-3)} = \frac{A}{x-1} + \frac{B}{x-3}$$

multiply both sides by $(x-1)(x-3)$

$$15x-31 = A(x-3)+B(x-1)$$
$$= (A+B)x + (-3A-B)$$

$$\Rightarrow \begin{array}{rl} 15 &= A+B \\ -31 &= -3A-B \\ \hline -16 &= -2A \end{array} \quad \Rightarrow A=8,\ B=7$$

$$\int_4^6 \frac{15x-31}{(x-1)(x-3)}\,dx = \int_4^6 \left(\frac{8}{x-1} + \frac{7}{x-3}\right) dx$$

$$= \Big[8\ln|x-1| + 7\ln|x-3|\Big]_4^6$$

$$= \big(8\ln|6-1| + 7\ln|6-3|\big) - \big(8\ln|4-1| + 7\ln|4-3|\big)$$

$$= 8\ln 5 + 7\ln 3 - 8\ln 3 - 7\underbrace{\ln 1}_{0}$$

$$= 8\ln 5 - \ln 3$$

Question 4. Use both the Trapezoid Rule and Simpson's Rule with $n = 4$ to estimate the definite integral.

$$\int_0^2 \frac{1}{x^3+1}\,dx$$

$[a,b] = [0,2] \quad n=4 \quad \Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{2}{4} = \frac{1}{2}$ or 0.5

$x_0 = 0 \quad x_1 = 0.5 = \frac{1}{2} \quad x_2 = 1 \quad x_3 = 1.5 = \frac{3}{2} \quad x_4 = 2$

$f(x_0) = f(0) = \frac{1}{0+1} = 1$

$f(x_1) = f(\frac{1}{2}) = \frac{1}{\frac{1}{8}+1} \cdot \frac{8}{8} = \frac{8}{1+8} = \frac{8}{9}$

$f(x_2) = f(1) = \frac{1}{1+1} = \frac{1}{2}$

$f(x_3) = f(\frac{3}{2}) = \frac{1}{\frac{27}{8}+1} \cdot \frac{8}{8} = \frac{8}{27+8} = \frac{8}{35}$

$f(x_4) = f(2) = \frac{1}{8+1} = \frac{1}{9}$

Trapezoid approximation: $\frac{1}{2}\Delta x \left(f(0) + 2f(0.5) + 2f(1) + 2f(1.5) + f(2)\right)$

$= \frac{1}{4}\left(1 + \frac{16}{9} + 1 + \frac{16}{35} + \frac{1}{9}\right) = \frac{1369}{1260} \approx 1.0865$

Simpson's approximation: $\frac{1}{3}\Delta x \left(f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + f(2)\right)$

$= \frac{1}{6}\left(1 + \frac{32}{9} + 1 + \frac{32}{35} + \frac{1}{9}\right) = \frac{691}{630} \approx 1.0968$

Recall $a^3+b^3 = (a+b)(a^2-ab+b^2)$

Therefore $x^3+1 = (x+1)(x^2-x+1)$

BONUS QUESTION: Can you find an antiderivative of $\frac{1}{x^3+1}$ by hand? It's time-consuming, but possible. Hint: First factor x^3+1 using the formula for factoring a sum of cubes. Then use partial fractions. Then do some more algebra.

Since it is *possible* but very *time-consuming* to find an antiderivative of $\frac{1}{x^3+1}$, you can see why we might want to do approximate integration instead, since then we just have to do messy arithmetic instead of messy algebra.

$$\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

multiply both sides by $(x+1)(x^2-x+1)$

$$1 = A(x^2-x+1) + (Bx+C)(x+1)$$

$$= Ax^2 - Ax + A + Bx^2 + Bx + Cx + C$$

$$\Rightarrow \left.\begin{aligned} 0 &= A+B \\ 0 &= -A+B+C \\ 1 &= A+C \end{aligned}\right\} \Rightarrow A = \frac{1}{3}, \; B = \frac{-1}{3}, \; C = \frac{2}{3}$$

$$\frac{1}{x^3+1} = \frac{1/3}{x+1} + \frac{-\frac{1}{3}x + \frac{2}{3}}{x^2-x+1} = \cancel{\frac{1}{3}\cdot\frac{1}{x+1} - \frac{1}{3}\cdot\frac{x}{x^2-x+1} + \frac{2}{3}\cdot\frac{1}{x^2-x+1}}$$

ignore

$$= \frac{1}{3}\cdot\frac{1}{x+1} - \frac{1}{3}\cdot\frac{x-2}{x^2-x+1}$$

$$= \frac{1}{3}\cdot\frac{1}{x+1} - \frac{1}{6}\cdot\frac{2x-4}{x^2-x+1}$$

$$= \frac{1}{3}\cdot\frac{1}{x+1} - \frac{1}{6}\cdot\left(\frac{2x-1}{x^2-x+1} - \frac{3}{x^2-x+1}\right)$$

$$= \frac{1}{3}\cdot\frac{1}{x+1} - \frac{1}{6}\cdot\frac{2x-1}{x^2-x+1} + \frac{1}{2}\cdot\frac{1}{x^2-x+1}$$

$$= \frac{1}{3}\cdot\frac{1}{x+1} - \frac{1}{6}\cdot\frac{2x-1}{x^2-x+1} + 2\cdot\frac{1}{4x^2-4x+4}$$

So

$$\int \frac{1}{x^3+1}\,dx = \frac{1}{3}\int \frac{1}{x+1}\,dx - \frac{1}{6}\underbrace{\int \frac{2x-1}{x^2-x+1}\,dx}_{\substack{\text{sub } u=x^2-x+1 \\ du=(2x-1)dx}} + \int \frac{2}{4x^2-4x+4}\,dx$$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln|x^2-x+1| + \int \frac{2}{4x^2-4x+4}\,dx$$

In this last integral, complete the square.

$$\int \frac{2}{4x^2-4x+1+3}\,dx = \int \frac{2}{(2x-1)^2+3}\,dx \qquad \text{Sub } 2x-1=\sqrt{3}\cdot u, \quad 2dx=\sqrt{3}\,du$$

$$= \int \frac{\sqrt{3}}{3u^2+3}\,du = \frac{1}{\sqrt{3}}\int \frac{1}{u^2+1}\,du = \frac{1}{\sqrt{3}}\arctan u$$

$$= \frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)$$

Final answer:

$$\int \frac{1}{x^3+1}\,dx = \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln|x^2-x+1| + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)$$

$$+C$$