WRITE YOUR NAME:

MAC 2312 WRITTEN HOMEWORK #3

Due Tuesday January 30th, in Canvas

Question 1. Find the <u>area</u> bounded by the graphs of $y = x^2$ and $y = \sqrt{x}$.

Intersection points? (Might able to "guess" intersection points by knowing the graphs of $y=x^2$ and $y=\sqrt{x}$ and thinking about special numbers like 0 and 1.) Algebraically: $x^2=\sqrt{x} \Rightarrow (x^2)^2 = (\sqrt{x})^2$

$\Rightarrow x^4 = x \Rightarrow x^4 - x = 0 \Rightarrow x(x^3-1)=0 \Rightarrow x=0$ or $x^3=1$, $x=1$

Top curve and bottom curve? Can use test input $x = 1/4$

$x^2 = (1/4)^2 = 1/16 \leftarrow$ smaller output

$\sqrt{x} = \sqrt{1/4} = 1/2 \leftarrow$ bigger output

Can also conclude this by knowing the shapes of the graphs.

$y=x^2$ $\quad$ $y=\sqrt{x}$

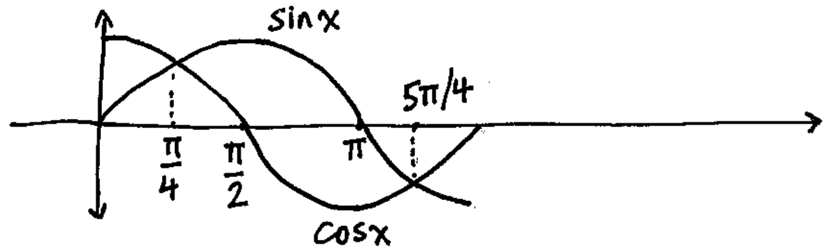
$$\text{Area} = \int_{x=0}^{x=1} \underbrace{(\underbrace{\text{top}}_{y=\sqrt{x}} - \underbrace{\text{bottom}}_{y=x^2})}_{\text{height}} \underbrace{dx}_{\text{width}} \rightarrow \int_0^1 (\sqrt{x} - x^2)\,dx$$

$$= \int_0^1 (x^{1/2} - x^2)\,dx = \left[\frac{x^{3/2}}{3/2} - \frac{x^3}{3}\right]_0^1 = \left[\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right]_0^1$$

$$= \left(\frac{2}{3} - \frac{1}{3}\right) - (0-0) = \frac{1}{3}$$

Question 2. Let R be the region bounded by the curves $y = \sin x$ and $y = \cos x$ on the interval $[\pi/4, 5\pi/4]$. Find the area of R.

Rough sketch:



If $\frac{\pi}{4} \leq x \leq \frac{5\pi}{4}$ then $\sin x \geqslant \cos x$

$$\text{Area} = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\,dx = \Big[-\cos x - \sin x\Big]_{\pi/4}^{5\pi/4}$$

$$= \Big[\cos x + \sin x\Big]_{5\pi/4}^{\pi/4} = \left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right) - \left(\cos\frac{5\pi}{4} + \sin\frac{5\pi}{4}\right)$$

$$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - \left(\frac{-\sqrt{2}}{2} + \frac{-\sqrt{2}}{2}\right)$$

$$= \sqrt{2} - (-\sqrt{2}) = \sqrt{2} + \sqrt{2}$$

$$= 2\sqrt{2}$$

Question 3. Find the area bounded by the graphs of $x = y^2 - 3y + 12$ and $x = -2y^2 - 6y + 30$.

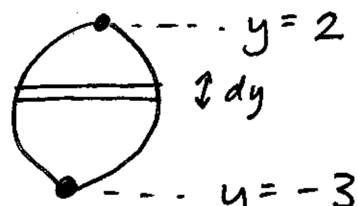
Intersection points? $y^2 - 3y + 12 = -2y^2 - 6y + 30$

$$3y^2 + 3y - 18 = 0$$

$$3(y^2 + y - 6) = 0$$

$$3(y+3)(y-2) = 0 \Rightarrow y = -3,\ y = 2$$

Rough picture:



Which is left curve, which is right curve?
Could plug in "test input" between $y=-3$ and $y=2$.
Say $y=0$. So $y^2-3y+12$ = small (left)
$-2y^2-6y+30$ = big (right)

$$\text{Area} = \int_{y=-3}^{y=2} (\text{right} - \text{left})\, dy = \int_{-3}^{2} \left(-2y^2 - 6y + 30 - (y^2 - 3y + 12)\right) dy$$

$$= \int_{-3}^{2} (-3y^2 - 3y + 18)\, dy = \left[-y^3 - \frac{3y^2}{2} + 18y\right]_{-3}^{2}$$

$$= -\left[y^3\right]_{-3}^{2} - \frac{3}{2}\left[y^2\right]_{-3}^{2} + 18\left[y\right]_{-3}^{2}$$

$$= \left[y^3\right]_{2}^{-3} + \frac{3}{2}\left[y^2\right]_{2}^{-3} + 18\left[y\right]_{-3}^{2}$$

$$= -27 - 8 + \frac{3}{2}(\underbrace{9-4}_{5}) + 18(\underbrace{2+3}_{5})$$

$$= -35 + \frac{15}{2} + 90$$

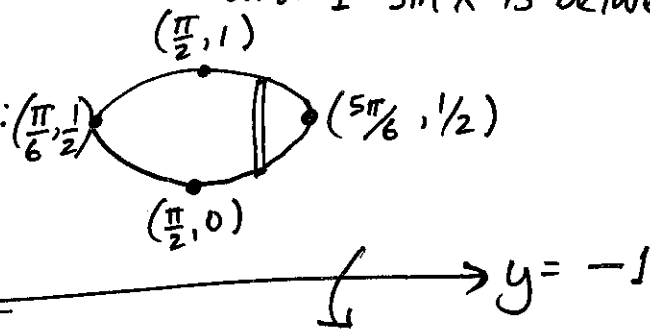
$$= 55 + \frac{15}{2}$$

$$= \frac{110}{2} + \frac{15}{2} = \frac{125}{2}$$

Question 4. Let A be the region bounded by the curves $y = \sin x$ and $y = 1 - \sin x$ on the interval $[\pi/6, 5\pi/6]$. Find the volume obtained when A is revolved around the line $y = -1$.

Note: If $\frac{\pi}{6} \le x \le \frac{5\pi}{6}$ then $\sin x$ is between $\frac{1}{2}$ and 1 and $1 - \sin x$ is between $\frac{1}{2}$ and 0

Rough picture:



Washers

$R = (\text{top curve}) - (\text{axis of rev.}) = \sin x - (-1) = \sin x + 1$

$r = (\text{bottom curve}) - (\text{axis of rev.}) = 1 - \sin x - (-1) = 2 - \sin x$

$$\text{Volume} = \int_{\pi/6}^{5\pi/6} (\pi R^2 - \pi r^2)\,dx = \pi \int_{\pi/6}^{5\pi/6} \left((\sin x + 1)^2 - (2 - \sin x)^2\right) dx$$

$$= \pi \int_{\pi/6}^{5\pi/6} \left(\sin^2 x + 2\sin x + 1 - (4 - 4\sin x + \sin^2 x)\right) dx$$

$$= \pi \int_{\pi/6}^{5\pi/6} (6\sin x - 3)\,dx = \pi \Big[-6\cos x - 3x\Big]_{\pi/6}^{5\pi/6}$$

$$= \pi \Big[6\cos x + 3x\Big]_{5\pi/6}^{\pi/6} = 6\pi \Big[\cos x\Big]_{5\pi/6}^{\pi/6} + 3\pi \Big[x\Big]_{5\pi/6}^{\pi/6}$$

$$= 6\pi\left(\frac{\sqrt{3}}{2} - \frac{-\sqrt{3}}{2}\right) + 3\pi \cdot \frac{-4\pi}{6} = 6\pi\sqrt{3} - 2\pi^2$$

or $2\pi(3\sqrt{3} - \pi)$