WRITE YOUR NAME:

MAC 2312 WRITTEN HOMEWORK #4

Due Tuesday February 6th, in Canvas

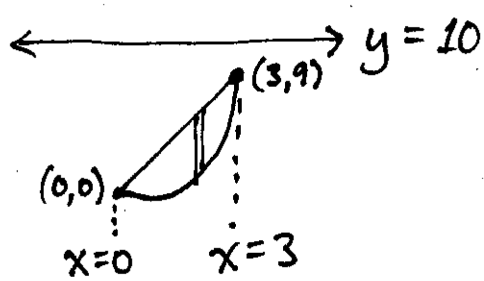
$y = f(x) \Rightarrow$

Question 1. Let A be the region bounded by the curves $y = x^2$ and $y = 3x$. Find the volume obtained when the region A is revolved around the line $y = 10$.

Intersections? $x^2 = 3x \Rightarrow x^2 - 3x = 0 \Rightarrow x(x-3) = 0$

$x = 0, \; x = 3$

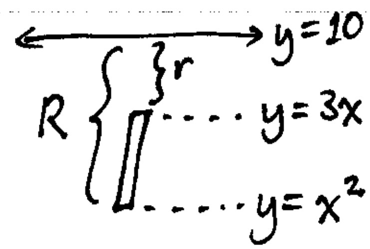
Picture:



$y = 3x$ is top curve

$y = x^2$ is bottom curve

Vertical slice around horizontal line → WASHERS



R = distance between "far curve" and axis of revolution

$= 10 - x^2$

r = distance between "near curve" and axis of revolution

$= 10 - 3x$

$$\text{Volume} = \int_{x=0}^{x=3} \left(\pi R^2 - \pi r^2\right) dx = \pi \int_0^3 \left(R^2 - r^2\right) dx$$

$$= \pi \int_0^3 \left((10 - x^2)^2 - (10 - 3x)^2 \right) dx$$

$$= \pi \int_0^3 \left(100 - 20x^2 + x^4 - (100 - 60x + 9x^2) \right) dx$$

$$= \pi \int_0^3 \left(100 - 20x^2 + x^4 - 100 + 60x - 9x^2\right) dx$$

$$= \pi \int_0^3 \left(x^4 + 60x - 29x^2\right) dx$$

$$= \pi \left[\frac{x^5}{5} + 30x^2 - 29\frac{x^3}{3} \right]_0^3$$

$$= \pi \left(\frac{3^5}{5} + 30 \cdot 3^2 - \underbrace{29 \cdot \frac{3^3}{3}}_{29 \cdot 3^2} \right)$$ ← CAN LEAVE IT LIKE THIS

$$= \pi \cdot 3^2 \cdot \left(\frac{3^3}{5} + \underbrace{30 - 29}_{1} \right)$$

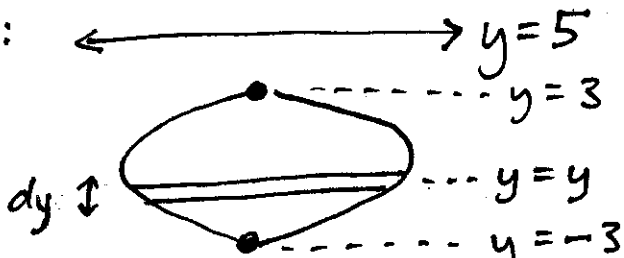
$$= \pi \cdot 3^2 \cdot \left(\frac{27}{5} + \frac{5}{5} \right) = \pi \cdot 9 \cdot \frac{32}{5} = \frac{288\pi}{5}$$

$\underset{\sim}{x} = f(y) \Rightarrow$ ▭ $\updownarrow dy$

Question 2. Let A be the region bounded by the curves $x = y^2$ and $x = 18 - y^2$. Find the volume obtained when the region A is revolved around the line $y = 5$.

Intersections? $y^2 = 18 - y^2 \Rightarrow 2y^2 = 18 \Rightarrow y^2 = 9 \Rightarrow y = -3, y = 3$

Rough picture:



Slice is PARALLEL to axis of revolution $\Rightarrow$ cylindrical SHELLS

Can choose "test input" between $y=-3$ and $y=3$, say $y=0$

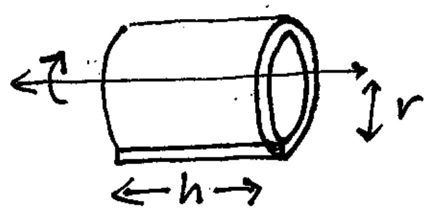
We find $x = y^2$ gives smaller outputs, $x = 18 - y^2$ gives bigger outputs

r = distance between typical slice and axis of revolution

$r = 5 - y$

h = distance from "bigger" curve to "smaller" curve = right minus left

$$h = \underbrace{18 - y^2}_{\text{right}} - \underbrace{y^2}_{\text{left}} = 18 - 2y^2$$



$$\text{Volume} = \int_{y=-3}^{y=3} 2\pi r h \, dy = 2\pi \int_{-3}^{3} \underbrace{(5-y)}_{r} \underbrace{(18-2y^2)}_{h} \, dy$$

$$= 2\pi \int_{-3}^{3} (90 - 18y - 10y^2 + 2y^3) \, dy$$

$$= 2\pi \cdot \left(\int_{-3}^{3} 90\,dy - \underbrace{\int_{-3}^{3} 18y\,dy}_{\substack{=0 \\ \text{(integral of} \\ \text{odd function} \\ \text{on } [-a,a])}} - \int_{-3}^{3} 10y^2\,dy + \underbrace{\int_{-3}^{3} 2y^3\,dy}_{\substack{=0 \\ \text{(integral of} \\ \text{odd function} \\ \text{on } [-a,a])}} \right)$$

We can also use the rule that $\int_{-a}^{a} f(y)\,dy = 2\int_{0}^{a} f(y)\,dy$

<u>if</u> f is an <u>even</u> function

$$\text{Volume} = 2\pi \cdot \left(2\int_{0}^{3} 90\,dy - 2\int_{0}^{3} 10y^2\,dy \right)$$

$$= 2\pi \cdot \left(2 \cdot \Big[90y \Big]_{0}^{3} - 2 \cdot \Big[\frac{10y^3}{3} \Big]_{0}^{3} \right)$$

$$= 2\pi \cdot \left(2 \cdot 90 \cdot 3 - 2 \cdot \underbrace{\frac{10 \cdot 3^3}{3}}_{= 10 \cdot 9} \right)$$

$$= 2\pi \cdot (540 - 180) = 2\pi \cdot 360 = 720\pi$$

Question 3. Find the length of the curve

$$y = 3\ln x - \frac{x^2}{24}$$

on the interval $[1, 6]$.

$$\frac{dy}{dx} = 3\cdot\frac{1}{x} - \frac{2x}{24} = \frac{3}{x} - \frac{x}{12}$$

$$\text{Length} = \int_{x=1}^{x=6} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_1^6 \sqrt{1+\left(\frac{3}{x}-\frac{x}{12}\right)^2}\,dx$$

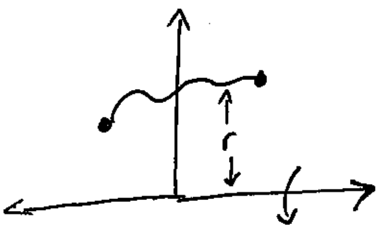
$$= \int_1^6 \sqrt{1+\frac{9}{x^2}\underbrace{-\frac{1}{2}}_{2\cdot\frac{3}{x}\cdot\frac{-x}{12}}+\frac{x^2}{144}}\,dx = \int_1^6 \sqrt{\frac{9}{x^2}+\frac{1}{2}+\frac{x^2}{144}}\,dx$$

$$= \int_1^6 \sqrt{\left(\frac{3}{x}+\frac{x}{12}\right)^2}\,dx = \int_1^6 \left(\frac{3}{x}+\frac{x}{12}\right)dx$$

$$= \left[3\ln|x| + \frac{x^2}{24}\right]_1^6 = 3\underbrace{\left[\ln|x|\right]_1^6}_{\ln 6 - \underbrace{\ln 1}_{0}} + \frac{1}{24}\underbrace{\left[x^2\right]_1^6}_{36-1}$$

$$= 3\ln 6 + \frac{35}{24}$$

Question 4 The portion of the curve $y = \sqrt{1-x^2}$ between $x = -1/2$ and $x = 1/2$ is revolved around the x-axis. Find the area of the surface generated.



$$\text{Surface area} = \int_{x=-1/2}^{x=1/2} 2\pi r\, ds$$

where $r = f(x)$ and $ds = \sqrt{1+(f'(x))^2}\,dx$

$$f(x) = \sqrt{1-x^2} = (1-x^2)^{1/2} \Rightarrow f'(x) = \frac{1}{2}(1-x^2)^{-1/2}\cdot(-2x) = \frac{-x}{\sqrt{1-x^2}}$$

$$\text{Surface area} = 2\pi\int_{-1/2}^{1/2} \underbrace{\sqrt{1-x^2}}_{r} \cdot \underbrace{\sqrt{1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\,dx}_{ds}$$

$$= 2\pi\int_{-1/2}^{1/2} \sqrt{1-x^2}\cdot\sqrt{\underset{\uparrow \atop \frac{1-x^2}{1-x^2}}{1} + \frac{x^2}{1-x^2}}\,dx$$

$$\frac{1-x^2}{1-x^2} + \frac{x^2}{1-x^2} = \frac{1}{1-x^2}$$

$$= 2\pi\int_{-1/2}^{1/2} \sqrt{1-x^2}\cdot\sqrt{\frac{1}{1-x^2}}\,dx$$

$$= 2\pi\int_{-1/2}^{1/2} 1\,dx = 2\pi\Big[x\Big]_{-1/2}^{1/2} = 2\pi\underbrace{\left(\frac{1}{2} - \frac{-1}{2}\right)}_{1}$$

$$= 2\pi$$