WRITE YOUR NAME:

MAC 2312 Test 1 Thursday February 8th
Total possible score: 18 points

Question 1. Evaluate the sum.

$$\sum_{k=1}^{7}(3k-3)$$

METHOD 1 : Directly.

$$\underbrace{(3\cdot1-3)}_{\substack{3-3\\=0}}+\underbrace{(3\cdot2-3)}_{\substack{6-3\\=3}}+\underbrace{(3\cdot3-3)}_{\substack{9-3\\=6}}+\underbrace{(3\cdot4-3)}_{\substack{12-3\\=9}}+\underbrace{(3\cdot5-3)}_{\substack{15-3\\=12}}+\underbrace{(3\cdot6-3)}_{\substack{18-3\\=15}}+\underbrace{(3\cdot7-3)}_{\substack{21-3\\=18}}$$

$0+3+6+9+12+15+18 \longrightarrow$ One possible trick:

$3+6+9+12+15+18$

$$=\underbrace{(3+18)}_{21}+\underbrace{(6+15)}_{21}+\underbrace{(9+12)}_{21} = \boxed{63}$$

METHOD 2:

$$\sum_{k=1}^{7}3k+\sum_{k=1}^{7}(-3) = 3\underbrace{\sum_{k=1}^{7}k}_{\underbrace{1+2+3+4+5+6+7}_{\text{Use a formula OR just add them}}} - \underbrace{\sum_{k=1}^{7}3}_{\underbrace{3+3+3+3+3+3+3}_{7\cdot3}}$$

$$=3\cdot28-7\cdot3$$

$$=84-21=\boxed{63}$$

Question 2. Evaluate the definite integral. You can use geometry if it helps.

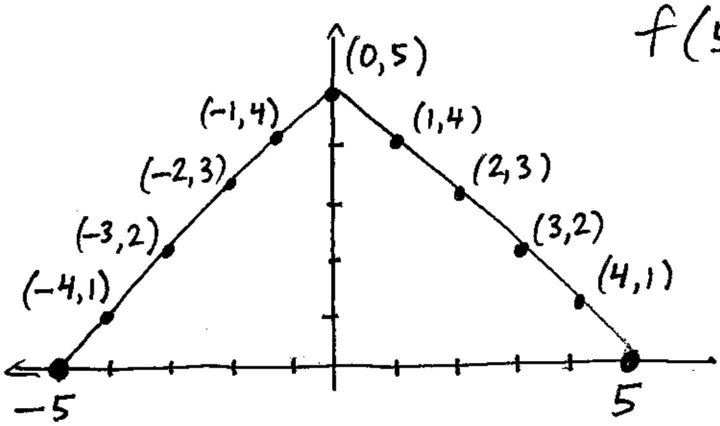
$$\int_{-5}^{5} (5 - |x|)\, dx$$

METHOD 1: Draw graph. If $f(x) = 5 - |x|$

then for example we have $f(-5) = 5 - |-5| = 5 - 5 = 0$

$f(0) = 5 - |0| = 5 - 0 = 5$

$f(5) = 5 - |5| = 5 - 5 = 0$



Integral = area of triangle with base 10 and height 5

$= \frac{1}{2} \cdot 10 \cdot 5 = \boxed{25}$

METHOD 2: If $x > 0$ then $|x| = x$ so $5 - |x| = 5 - x$

If $x < 0$ then $|x| = -x$ so $5 - |x| = 5 - (-x) = 5 + x$

$$\text{Integral} = \int_{-5}^{0} (5 - |x|)\, dx + \int_{0}^{5} (5 - |x|)\, dx$$

$$= \int_{-5}^{0} (5 + x)\, dx + \int_{0}^{5} (5 - x)\, dx$$

$$= \left[5x + \frac{x^2}{2}\right]_{-5}^{0} + \left[5x - \frac{x^2}{2}\right]_{0}^{5}$$

$$= (0 + 0) - \left(-25 + \frac{25}{2}\right) + \left(25 - \frac{25}{2}\right) - (0 - 0)$$

$$= 25 - \frac{25}{2} + 25 - \frac{25}{2} = 50 - \frac{50}{2} = 50 - 25 = \boxed{25}$$

Question 3. Evaluate the definite integral.

$$\int_1^2 \left(8x^3 + \frac{8}{x^3}\right) dx$$

$$= \int_1^2 \left(8x^3 + 8x^{-3}\right) dx$$

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$
if n is any constant EXCEPT -1.
If $n = -3$
then $n+1 = -3+1 = -2$.

$$= \left[8 \cdot \frac{x^4}{4} + 8 \cdot \frac{x^{-2}}{-2} \right]_1^2$$

$$= \left[2x^4 - 4x^{-2} \right]_1^2$$

$$= \left[2x^4 - \frac{4}{x^2} \right]_1^2$$

$$= 2\left[x^4\right]_1^2 - 4\left[\frac{1}{x^2}\right]_1^2$$

$$= 2\left[x^4\right]_1^2 + 4\left[\frac{1}{x^2}\right]_2^1$$

$$= 2(\underbrace{2^4 - 1^4}_{16-1}) + 4(\underbrace{\tfrac{1}{1} - \tfrac{1}{4}}_{3/4})$$

$$= 2 \cdot 15 + 4 \cdot \frac{3}{4} = 30 + 3 = \boxed{33}$$

Question 4. Evaluate the indefinite integral.

$$\int \frac{1}{x^3} \cos\left(\frac{1}{x^2}\right) dx = \int x^{-3} \cos(x^{-2})\, dx$$

Try substituting $u = \dfrac{1}{x^2} = x^{-2}$

Then $\dfrac{du}{dx} = -2x^{-3} \Rightarrow du = -2x^{-3} dx$

$$\Rightarrow \underbrace{-\frac{1}{2} du}_{} = \underbrace{x^{-3} dx}_{}$$

$$\text{Integral} = \int \cos(\underbrace{x^{-2}}_{u}) \cdot \underbrace{x^{-3} dx}_{}$$

$$= \int \cos(u) \cdot \underbrace{\frac{-1}{2} du}_{} = -\frac{1}{2} \int \cos u\, du$$

$$= -\frac{1}{2} \sin u + C = \boxed{-\frac{1}{2} \sin\left(\frac{1}{x^2}\right) + C}$$

Question 5. Evaluate the definite integral.

$$\int_1^7 \frac{2x}{\sqrt{x^2+15}}\,dx$$

Try substituting $u = x^2 + 15$

$$du = 2x\,dx \qquad \frac{du}{dx} = 2x$$

If $x = 1$ then $u = 1^2 + 15 = 1 + 15 = 16$

If $x = 7$ then $u = 7^2 + 15 = 49 + 15 = 64$

$$\int_{x=1}^{x=7} \frac{1}{\sqrt{\underbrace{x^2+15}_{u}}} \cdot 2x\,dx = \int_{u=16}^{u=64} \frac{1}{\sqrt{u}} \cdot du$$

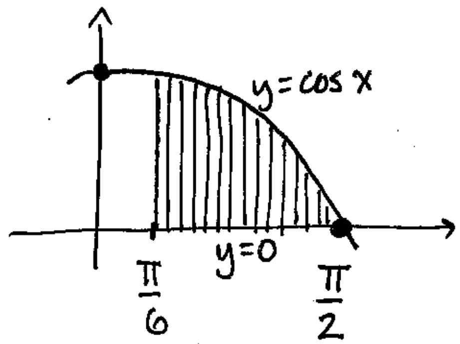
$$= \int_{u=16}^{u=64} u^{-1/2}\,du = \left[\frac{u^{1/2}}{1/2}\right]_{u=16}^{u=64}$$

$$= 2\left[u^{1/2}\right]_{u=16}^{u=64} = 2\left(64^{1/2} - 16^{1/2}\right)$$

$$= 2(8-4) = 2\cdot 4 = \boxed{8}$$

Question 6. Find the **area** bounded by the curves $y = \cos x$ and $y = 0$ between $x = \pi/6$ and $x = \pi/2$.

We know the behavior of $\cos x$ so we can sketch a graph:



On the interval $\frac{\pi}{6} \le x \le \frac{\pi}{2}$

$y = \cos x$ is the top curve
and $y = 0$ is the bottom curve

$$\text{Area} = \int_{\pi/6}^{\pi/2} (\cos x - 0)\, dx = \int_{\pi/6}^{\pi/2} \cos x \, dx$$

$$= \Big[\sin x \Big]_{\pi/6}^{\pi/2} = \underbrace{\sin\frac{\pi}{2}}_{1} - \underbrace{\sin\frac{\pi}{6}}_{1/2} = 1 - \frac{1}{2} = \boxed{\frac{1}{2}}$$

θ	$\sin\theta$
0	0
$\pi/6$	$\sqrt{1}/2 = 1/2$
$\pi/4$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$
$\pi/2$	1

$\}$ Note the pattern here

Question 7. Evaluate the definite integral.

$$\int_4^9 \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$$

Try substituting $u = \sqrt{x} = x^{1/2}$

$$\frac{du}{dx} = \frac{1}{2} x^{-1/2}$$

$$du = \frac{1}{2} x^{-1/2}\, dx$$

$$2du = x^{-1/2} dx$$

$$2du = \frac{1}{\sqrt{x}}\, dx$$

If $x = 4$ then $u = \sqrt{4} = 2$

If $x = 9$ then $u = \sqrt{9} = 3$

$$\int_{x=4}^{x=9} e^{\sqrt{x}} \cdot \frac{1}{\sqrt{x}}\, dx = \int_{u=2}^{u=3} e^{u} \cdot 2du$$

$$= 2\int_{u=2}^{u=3} e^{u}\, du = 2\left[e^{u}\right]_{u=2}^{u=3}$$

$$= \boxed{2(e^3 - e^2)}$$

If you like, you can factor this as $2e^2(e-1)$

Question 8. Let A be the region bounded by $\overbrace{y = 2x^2 - x^3}^{y=f(x)}$ and the $\underbrace{x\text{-axis}}_{y=0}$. Find the **volume** obtained by revolving A around the y-axis. $\{y = g(x)\}$

$y = f(x) \rightarrow$ Vertical slices. Top curve and bottom curve.

dx

Intersections of the two curves? $2x^2 - x^3 = 0$

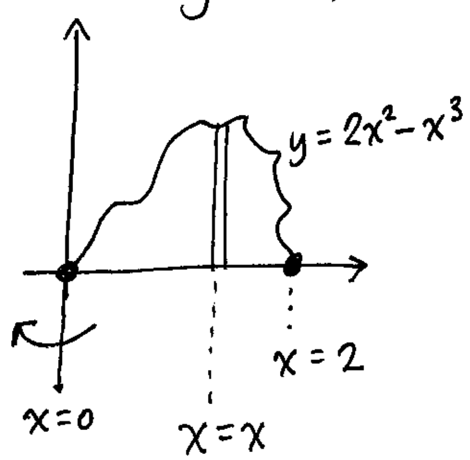
$\Rightarrow x^2(2-x) = 0 \quad \Rightarrow x = 0, x = 2$

Which is top, which is bottom? Test input could be $x = 1$

$y = 2 \cdot 1^2 - 1^3 = 2 - 1 =$ positive, so this is top curve.

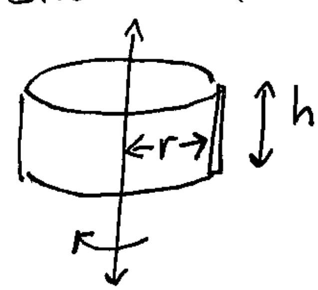
Given: Revolving around y-axis, i.e. revolving around $x = 0$.

Rough picture:



Slice is PARALLEL to axis

$\Rightarrow$ SHELLS $(2\pi r h)$



Here $h =$ (top curve) $-$ (bottom curve) $= 2x^2 - x^3 - 0 = 2x^2 - x^3$

$r =$ distance from typical slice to axis of revolution $= x - 0 = x$

$$\text{Volume} = 2\pi \int_0^2 \underbrace{x}_{r} \underbrace{(2x^2 - x^3)}_{h}\, dx = 2\pi \int_0^2 (2x^3 - x^4)\, dx$$

$$= 2\pi \left[\frac{2x^4}{4} - \frac{x^5}{5} \right]_0^2 = 2\pi \left(\frac{32}{4} - \frac{32}{5} \right) = 2\pi \cdot 32 \cdot \underbrace{\left(\frac{1}{4} - \frac{1}{5} \right)}_{\frac{5}{20} - \frac{4}{20}}$$

$$= 2\pi \cdot 32 \cdot \frac{1}{20} = 2\pi \cdot 8 \cdot \frac{1}{5} = 16\pi/5$$

$y=f(x)$ $\quad$ dx

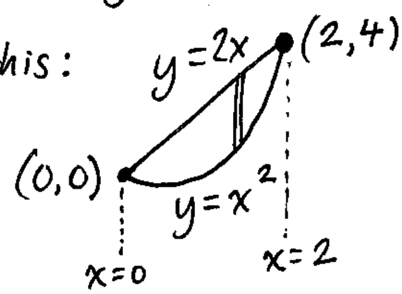
Question 9. Let A be the region bounded by $y = x^2$ and $y = 2x$. Find the **volume** obtained by revolving A around the line $y = -2$.

$y=f(x)$ → vertical slices, top curve and bottom curve

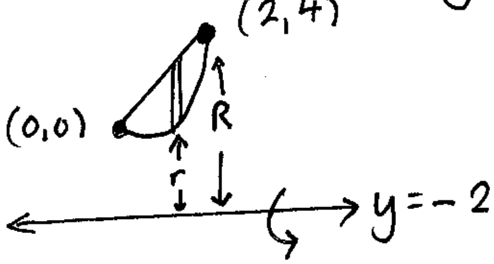
Intersections of the two curves? $x^2 = 2x \Rightarrow x^2 - 2x = 0$

$\Rightarrow x(x-2) = 0 \Rightarrow x=0,\ x=2$

Which is top, which is bottom? Could use test input, say $x=1$.
Also, we just "know" the shapes of $y=x^2$ and $y=2x$.
Region looks something like this:



Given: We're revolving around $y=-2$, which is a horizontal line



Slice is PERPENDICULAR to axis of revolution
$\Rightarrow$ WASHERS $(\pi R^2 - \pi r^2)$

R = distance from top curve to axis of revolution
$= 2x - (-2) = 2x + 2$

r = dist. from bottom curve to axis of rev. $= x^2 - (-2)$
$= x^2 + 2$

$$\text{Volume} = \pi \int_0^2 \left((\underbrace{2x+2}_{R})^2 - (\underbrace{x^2+2}_{r})^2 \right) dx$$

$$= \pi \int_0^2 \left(4x^2 + 8x + 4 \underbrace{- (x^4 + 4x^2 + 4)}_{= -x^4 - 4x^2 - 4} \right) dx = \pi \int_0^2 (8x - x^4)\, dx$$

$$= \pi \left[4x^2 - \frac{x^5}{5} \right]_0^2 = \pi \left(16 - \frac{32}{5} \right) = \pi \cdot 16 \cdot \left(1 - \frac{2}{5} \right) = \pi \cdot 16 \cdot \frac{3}{5}$$

$$= 48\pi/5$$